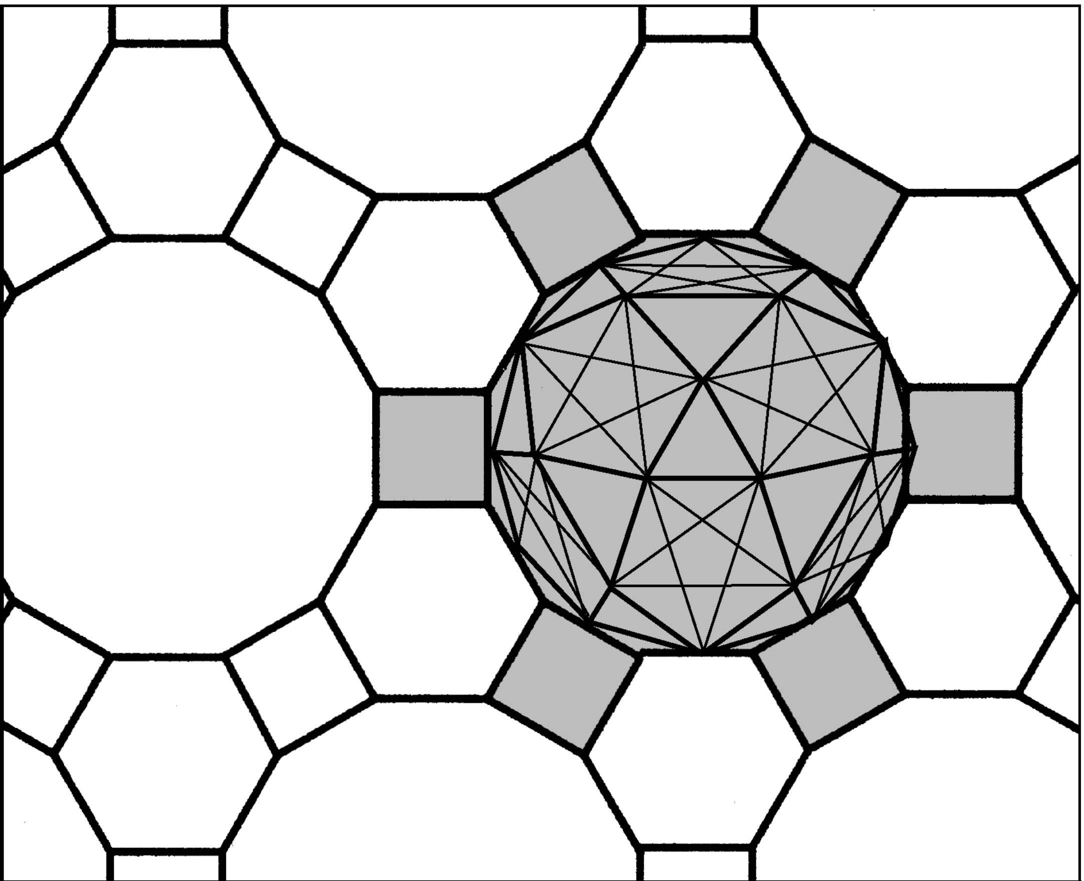
Workbook 2



Basic Functions

HELM

HELM: Helping Engineers Learn Mathematics

http://helm.lboro.ac.uk

About the HELM Project

HELM (Helping Engineers Learn Mathematics) materials were the outcome of a three-year curriculum development project undertaken by a consortium of five English universities led by Loughborough University, funded by the Higher Education Funding Council for England under the Fund for the Development of Teaching and Learning for the period October 2002 – September 2005, with additional transferability funding October 2005 – September 2006.

HELM aims to enhance the mathematical education of engineering undergraduates through flexible learning resources, mainly these Workbooks.

HELM learning resources were produced primarily by teams of writers at six universities: Hull, Loughborough, Manchester, Newcastle, Reading, Sunderland.

HELM gratefully acknowledges the valuable support of colleagues at the following universities and colleges involved in the critical reading, trialling, enhancement and revision of the learning materials:
Aston, Bournemouth & Poole College, Cambridge, City, Glamorgan, Glasgow, Glasgow Caledonian, Glenrothes Institute of Applied Technology, Harper Adams, Hertfordshire, Leicester, Liverpool, London Metropolitan, Moray College, Northumbria, Nottingham, Nottingham Trent, Oxford Brookes, Plymouth, Portsmouth, Queens Belfast, Robert Gordon, Royal Forest of Dean College, Salford, Sligo Institute of Technology, Southampton, Southampton Institute, Surrey, Teesside, Ulster, University of Wales Institute Cardiff, West Kingsway College (London), West Notts College.

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Production of this 2015 edition, containing corrections and minor revisions of the 2008 edition, was funded by the **sigma** Network.

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Contents 2

Basic Functions

Learning outcomes

In this Workbook you will learn about some of the basic building blocks of mathematics. You will gain familiarity with functions and variables. You will learn how to graph a function and what is meant by an inverse function. You will learn how to use a parametric approach to describe a function. Finally, you will meet some of the functions which occur in engineering and science: polynomials, rational functions, the modulus function and the unit step function.

Basic Concepts of Functions

2.1

Introduction

In engineering there are many quantities that change their value as time changes. For example, the temperature of a furnace may change with time as it is heated. Similarly, there are many quantities that change their value as the location of a point of interest changes. For example, the shear stress in a bridge girder will vary from point to point across the bridge. A quantity whose value can change is known as a **variable**. We use **functions** to describe how one variable changes as a consequence of another variable changing. There are many different types of function that are used by engineers. We will be examining some of these in later Sections. The purpose of this Section is to look at the basic concepts associated with all functions.

Prerequisites

Before starting this Section you should ...

- have a thorough understanding of basic algebraic notation and techniques

Learning Outcomes

On completion you should be able to ...

- explain what is meant by a function
- use common notations for functions
- explain what is meant by the argument of a function

1. The function rule

A function can be thought of as a **rule** which operates on an **input** and produces an **output**. This is often illustrated pictorially in two ways as shown in Figure 1. The first way is by using a **block diagram** which consists of a box showing the input, the output and the rule. We often write the rule inside the box. The second way is to use two sets, one to represent the input and one to represent the output with an arrow showing the relationship between them.

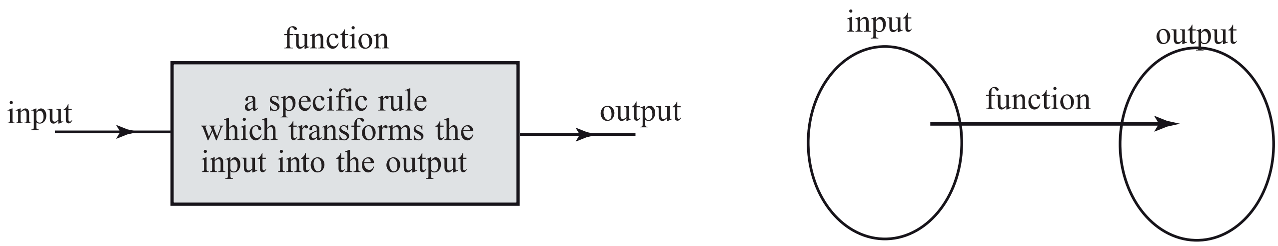


Figure 1: A general function

More precisely, a rule is a function if it produces only a **single** output for any given input. The function with the rule 'treble the input' is shown in Figure 2.

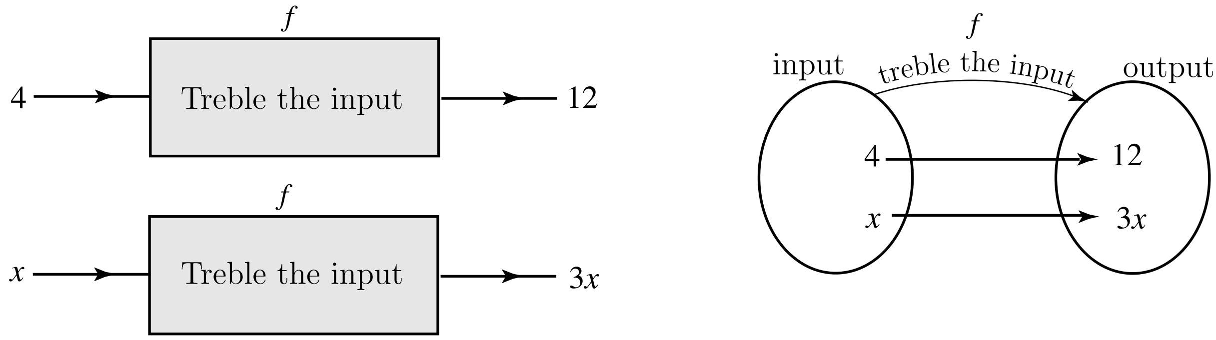


Figure 2: The function with the rule 'treble the input'

Note that with an input of 4 the function will produce an output of 12. With a more general input, x say, the output will be $3x$. It is usual to assign a letter or other symbol to a function in order to label it. The trebling function in Figure 2 has been given the symbol f.

Key Point 1

A function is a rule which operates on an input and produces a **single** output from that input.

Write down the output from the function shown in Figure 3 when the input is
(a) 4, (b) -3, (c) x (d) t.

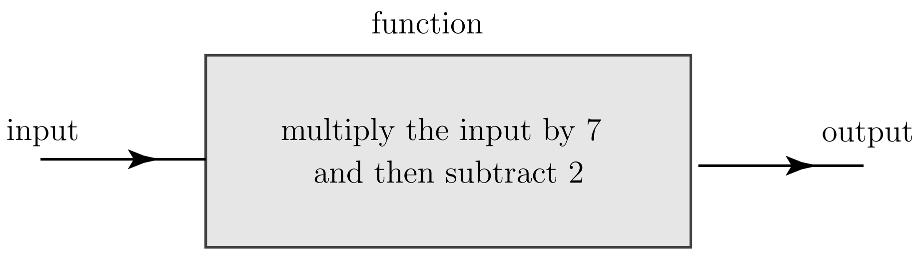


Figure 3

Your solution

In each case the function rule instructs you to multiply the input by 7 and then subtract 2. Evaluate the corresponding outputs.

Answer

(a) When the input is 4 the output is 26

(b) When the input is -3 the output is -23

(c) When the input is x the output is $7x - 2$

(d) When the input is t the output is $7t - 2$.

Several different notations are used by engineers to describe functions. For the trebling function in Figure 2 it is common to write

$$f(x) = 3x$$

This indicates that with an input x, the function, f, produces an output of $3x$. The input to the function is placed in the brackets after the 'f'. $f(x)$ is read as 'f is a function of x', or simply 'f of x', meaning that the value of the output from the function depends upon the value of the input x. The value of the output is often called the 'value of the function'.

Example 1

State in words the rule defined by each of the following functions:
(a) $f(x) = 6x$
(b) $f(t) = 6t - 1$
(c) $g(x) = x^2 - 7$
(d) $h(t) = t^3 + 5$
(e) $p(x) = x^3 + 5$

Solution

(a) The rule for f is 'multiply the input by 6'.

(b) Here the input has been labelled t. The rule for f is 'multiply the input by 6 and subtract 1'.

(c) Here the function has been labelled g. The rule for g is 'square the input and subtract 7'.

(d) The rule for h is 'cube the input and add 5'.

(e) The rule for p is 'cube the input and add 5'.

Note from Example 1, parts (d) and (e), that it is the rule that is important when describing a function and not the letters used. Both $h(t)$ and $p(x)$ instruct us to 'cube the input and add 5'.

Task

Write down a mathematical function which can be used to describe the following rules:
(a) 'square the input and divide the result by 2'. Use the letter x for input and the letter f to represent the function.
(b) 'divide the input by 3 and add 7'. Call the function g and call the input t.

Your solution

Answer

(a) $f(x) = \dfrac{x^2}{2}$, (b) $g(t) = \dfrac{t}{3} + 7$

Exercise

State the rule of each of the following functions:
(a) $f(x) = 5x$, (b) $f(t) = 5t$, (c) $f(x) = 8x + 10$, (d) $f(t) = 7t - 27$, (e) $f(t) = 1 - t$,

(f) $h(t) = \dfrac{t}{3} + \dfrac{2}{3}$, (g) $f(x) = \dfrac{1}{1 + x}$

Answers

(a) multiply the input by 5. (b) same as (a). (c) multiply the input by 8 and then add 10. (d) multiply the input by 7 and then subtract 27. (e) subtract the input from 1. (f) divide the input by 3 and then add 2/3. (g) add 1 to the input and then find the reciprocal of the result.

2. The argument of a function

The input to a function is sometimes called its **argument**. It is frequently necessary to obtain the output from a function if we are given its argument. For example, given the function $g(t) = 3t + 2$ we may require the value of the output when the argument is 4. We write this as $g(t = 4)$ or more usually and compactly as $g(4)$. In this case the value of $g(4)$ is $3 \times 4 + 2 = 14$.

Example 2

Given the function $f(x) = 3x + 1$ find
(a) $f(2)$
(b) $f(-1)$
(c) $f(6)$

Solution

(a) The output from the function needs to be found when the argument or input is 2. We need to replace x by 2 in the expression for the function. We find

$$f(2) = 3 \times 2 + 1 = 7$$

(b) Here the argument is -1. We find

$$f(-1) = 3 \times (-1) + 1 = -2$$

(c) $f(6) = 3 \times 6 + 1 = 19$.

Task

Given the function $g(t) = 6t + 4$ find (a) $g(3)$, (b) $g(6)$, (c) $g(-2)$

Your solution

Answer

a) $g(3) = 6 \times 3 + 4 = 22$, (b) $g(6) = 40$, (c) $g(-2) = -8$

It is possible to obtain the value of a function when the argument is an algebraic expression. Consider the following Example.

Example 3

Given the function $y(x) = 3x + 2$ find
(a) $y(t)$
(b) $y(2t)$
(c) $y(z + 2)$
(d) $y(5x)$

Solution

The rule for this function is 'multiply the input by 3 and then add 2'. We can apply this rule whatever the argument.

(a) In this case the argument is t. Multiplying this by 3 and adding 2 we find $y(t) = 3t + 2$. Equivalently we can replace x by t in the expression for the function, so, $y(t) = 3t + 2$.

(b) In this case the argument is $2t$. We need to replace x by $2t$ in the expression for the function. So $y(2t) = 3(2t) + 2 = 6t + 2$

(c) In this case the argument is $z + 2$. We find $y(z + 2) = 3(z + 2) + 2 = 3z + 8$. It is important to note that $y(z + 2)$ is **not** $y \times (z + 2) = yz + y2$ but instead reads 'y of $(z + 2)$' where 'of' means 'take the function of'.

(d) Here we have a complication. The argument is $5x$ and so there appears to be a clash of notation with the original expression for the function. There is no problem if we remember that the rule is to multiply the input by 3 and then add 2. The input now is $5x$. So $y(5x) = 3(5x) + 2 = 15x + 2$.

Task

Given the function $g(x) = 8 - 2x$ find (a) $g(4)$, (b) $g(4t)$, (c) $g(2x - 3)$

Your solution

(a)

(b)

(c)

Answer

(a) $g(4) = 8 - 2 \times 4 = 0$

(b) $g(4t) = 8 - 2 \times 4t = 8 - 8t$

(c) $g(2x - 3) = 8 - 2(2x - 3) = 14 - 4x$

Exercises

1. Explain what is meant by the 'argument' of a function.
2. Given the function $g(t) = 8t + 3$ find (a) $g(7)$, (b) $g(2)$, (c) $g(-0.5)$, (d) $g(-0.11)$
3. Given the function $f(t) = 2t^2 + 4$ find: (a) $f(x)$ (b) $f(2x)$ (c) $f(-x)$ (d) $f(4x + 2)$ (e) $f(3t + 5)$ (f) $f(\lambda)$ (g) $f(t - \lambda)$ (h) $f(\dfrac{t}{\alpha})$
4. Given $g(x) = 3x^2 - 7$ find (a) $g(3t)$, (b) $g(t + 5)$, (c) $g(6t - 4)$, (d) $g(4x + 9)$
5. Calculate $f(x + h)$ when (a) $f(x) = x^2$, (b) $f(x) = x^3$, (c) $f(x) = \dfrac{1}{x}$. In each case write down the corresponding expression for $f(x + h) - f(x)$.
6. If $f(x) = \dfrac{1}{(1 - x)^2}$ find $f(\dfrac{x}{\ell})$.

Answers

1. The argument is the input.
2. (a) 59, (b) 19, (c) -1, (d) 2.12.
3. (a) $2x^2 + 4$, (b) $8x^2 + 4$, (c) $2x^2 + 4$, (d) $32x^2 + 32x + 12$, (e) $18t^2 + 60t + 54$, (f) $2\lambda^2 + 4$, (g) $2(t - \lambda)^2 + 4$, (h) $\dfrac{2t^2}{\alpha^2} + 4$.
4. (a) $27t^2 - 7$, (b) $3t^2 + 30t + 68$, (c) $108t^2 - 144t + 41$, (d) $48x^2 + 216x + 236$.
5. (a) $x^2 + 2xh + h^2$, (b) $x^3 + 3x^2h + 3xh^2 + h^3$, (c) $\dfrac{1}{x + h}$.
   The corresponding expressions are (a) $2xh + h^2$, (b) $3x^2h + 3xh^2 + h^3$,
   (c) $\dfrac{1}{x + h} - \dfrac{1}{x} = -\dfrac{h}{x(x + h)}$.
6. $\dfrac{1}{(1 - \frac{x}{\ell})^2}$.

3. Composition of functions

Consider the two functions $g(x) = x^2$, and $h(x) = 3x + 5$. Block diagrams showing the rules for these functions are shown in Figure 4.

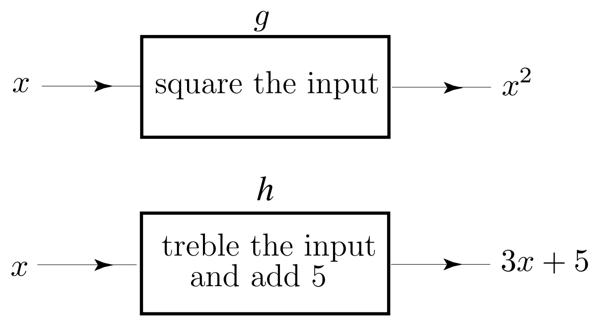


Figure 4: Block diagrams of two functions g and h

Suppose we place these Block diagrams together in series as shown in Figure 5, so that the output from function g is used as the input to function h.

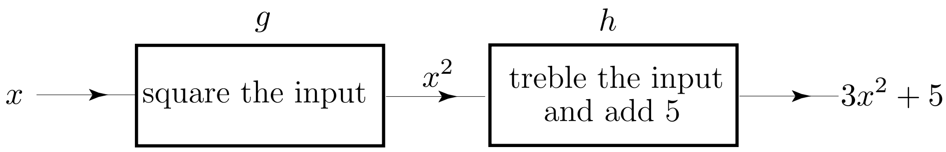


Figure 5: The composition of the two functions to give $h(g(x))$

Study Figure 5 carefully and deduce that when the input to g is x the output from the two functions in series is $3x^2 + 5$. Since the output from g is used as input to h we write

$$h(g(x)) = h(x^2) = 3x^2 + 5$$

The form $h(g(x))$ is known as the **composition** of the functions g and h.
Suppose we interchange the two functions so that h is applied first as shown in Figure 6.

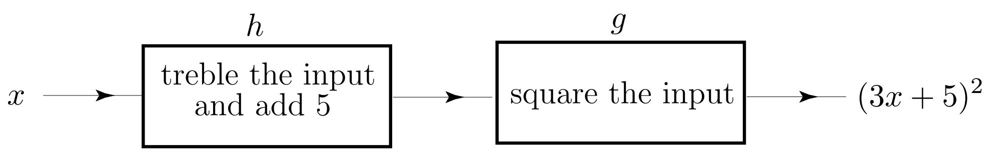


Figure 6: The composition of the two functions to give $g(h(x))$

Study Figure 6 and note that when the input to h is x the final output is $(3x + 5)^2$. We write

$$g(h(x)) = (3x + 5)^2$$

Note that the function $h(g(x))$ is different from $g(h(x))$.

Example 4

Given two functions $g(t) = 3t + 2$ and $h(t) = t + 3$ obtain an expression for the composition $g(h(t))$.

Solution

We have $g(h(t)) = g(t + 3)$. Now the rule for g is 'triple the input and add 2', and so we can write $g(t + 3) = 3(t + 3) + 2 = 3t + 11$ so, $g(h(t)) = 3t + 11$.

Task

Given the two functions $g(t) = 3t + 2$ and $h(t) = t + 3$ as in Example 4 above, obtain an expression for the composition $h(g(t))$.

Your solution

We have

$$h(g(t)) = h(3t + 2)$$

State the rule for h and write down $h(g(t))$.

Answer

'add 3 to the input', $h(3t + 2) = 3t + 5$. Note that $h(g(t)) \neq g(h(t))$.

Exercises

1. Find $f(g(x))$ when $f(x) = x - 7$ and $g(x) = x^2$.
2. If $f(x) = 8x + 2$ find $f(f(x))$.
3. If $f(x) = x + 6$ and $g(x) = x^2 - 5$ find (a) $f(g(0))$, (b) $g(f(0))$, (c) $g(g(2))$, (d) $f(g(7))$.
4. If $f(x) = \dfrac{x-3}{x+1}$ and $g(x) = \dfrac{1}{x}$ find $g(f(x))$.

Answers

1. $x^2 - 7$.
2. $8(8x + 2) + 2 = 64x + 18$.
3. (a) 1, (b) 31, (c) -4, (d) 50.
4. $\dfrac{x+1}{x-3}$.

Graphs of Functions and Parametric Form

2.2

Introduction

Engineers often find mathematical ideas easier to understand when these are portrayed visually as opposed to algebraically. Graphs are a convenient and widely-used way of portraying functions. By inspecting a graph it is easy to describe a number of properties of a function. For example, where is the function positive, and where is it negative? Where is it increasing and where is it decreasing? Do function values repeat? Questions like these can be answered once the graph of a function has been drawn. In this Section we will describe how the graph of a function is obtained and introduce various terminology associated with graphs.

We have seen in Section 2.1 that it is possible to represent a function using the form $y = f(x)$. An alternative representation is to write expressions for both y and x in terms of a third variable known as a **parameter**. The variables t or θ are normally used to denote the parameter.
For example, when a projectile such as a ball or rocket is thrown or launched, the x and y coordinates of its path can be described by a function in the form $y = f(x)$. However, it is often useful to also give its x coordinate as a function of the time after launch, that is $x(t)$, and the y coordinate similarly as $y(t)$. Here time t is the parameter.

Prerequisites

Before starting this Section you should . . .

- understand what is meant by a function

Learning Outcomes

On completion you should be able to . . .

- draw the graphs of a variety of functions
- explain what is meant by the domain and range of a function

1. The graph of a function

Consider the function $f(x) = 2x$. The output is obtained by multiplying the input by 2. We can choose several values for the input to this function and calculate the corresponding outputs. We have done this for integer values of x between -2 and 2 and the results are shown in Table 1.

Table 1

input, x	-2	-1	0	1	2
output, $f(x)$	-4	-2	0	2	4

To construct the graph of this function we first draw a pair of **axes** - a vertical axis and a horizontal axis. These are drawn at right-angles to each other and intersect at the **origin** as shown in Figure 7.

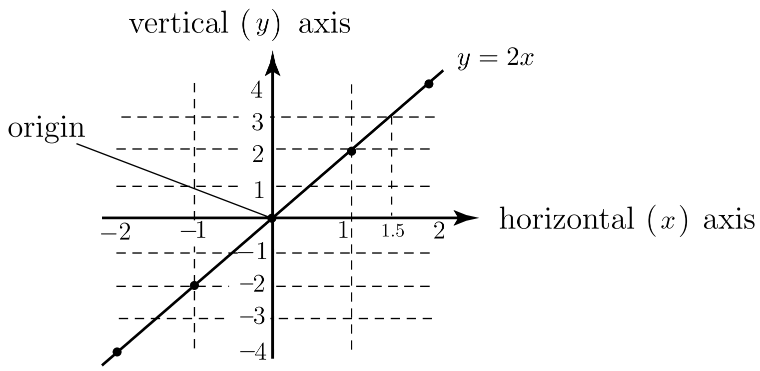


Figure 7: The two axes intersect at the origin

Each pair of input and output values can be represented on a graph by a single point. The input values are measured along the horizontal axis and the output values are measured along the vertical axis. The horizontal axis is often called the x axis. The vertical axis is commonly referred to as the y axis so that we often write the function as

$$y = f(x) = 2x$$

or simply

$$y = 2x$$

Each pair of x and y values in the table is plotted as a single point, shown as • in Figure 7. A general point is often labelled as (x, y). The values x and y are said to be the **coordinates** of the point. The points are then joined with a smooth curve to produce the required graph as shown in Figure 7. Note that in this case the graph is a straight line. The graph can then be used to find function values other than those given in the table. For example, directly from the graph we can see that when $x = 1.5$, the value of y is 3.

Task

Draw up a table of values of the function $f(x) = x^3$ for x between -3 and 3. Use the table to plot a graph of this function.

Complete the following table:

Your solution

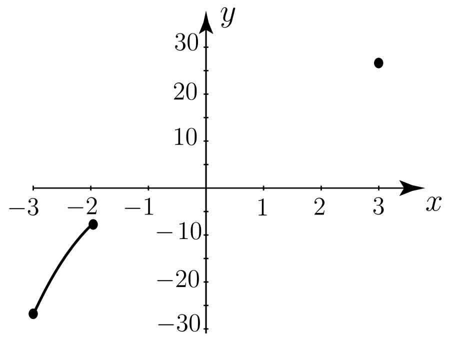
input, x	-3	-2	-1	0	1	2	3
output, $f(x)$	-27	-8					27

Answer

input, x	-3	-2	-1	0	1	2	3
output, $f(x)$	-27	-8	-1	0	1	8	27

Now add your points to the graph of $f(x) = x^3$ and draw a smooth curve through them:

Your solution



Dependent and independent variables

Since x and y can have a number of different values they are variables. Here x is called the **independent variable** and y is called the **dependent variable**. Knowing or choosing a value of the independent variable x, the function rule enables us to calculate the corresponding value of the dependent variable y. To show this dependence we often write $y(x)$. This is read as 'y is a function of x' or 'y depends upon x', or simply 'y of x'. Note that it is the independent variable which is the input to the function and the dependent variable which is the output.

The domain and range of a function

The set of values which we allow the independent variable to take is called the **domain** of the function. A domain is often an interval on the x axis. For example, the function

$$y = g(x) = 5x + 2, \qquad -5 \leq x \leq 20$$

has any value of x between -5 and 20 inclusive as its domain because it has been stated as this. If the domain of a function is not stated then it is taken to be the largest set possible. For example

$$h(t) = t^2 + 1$$

has domain $-\infty < x < \infty$ since h is defined for every value of t and the domain has not been stated otherwise.

Later, you will meet some functions for which certain values of the independent variable must be excluded from the domain because at these values the function would be undefined. One such example is $f(x) = \dfrac{1}{x}$ for which we must exclude the value $x = 0$, since $\dfrac{1}{0}$ is a meaningless quantity. Similarly, we must exclude the value $x = 2$ from the domain of $f(x) = \dfrac{1}{x-2}$.

The set of values of the function for a given domain, that is, the set of y values, is called the **range** of the function. The range of $g(x)$ (above) is $-23 \leq g(x) \leq 102$ and the range of $h(t)$ is $1 \leq h(t) < \infty$, although this may not be apparent to you at this stage. Usually the range of a function can be identified quite easily by inspecting its graph.

Example 5

Consider the function given by $g(t) = 2t^2 + 1$, $-2 \leq t \leq 2$.

(a) State the domain of the function.

(b) Plot a graph of the function.

(c) Deduce the range of the function from the graph.

Solution

(a) The domain is given as the interval $-2 \leq t \leq 2$, that is any value of t between -2 and 2 inclusive.

(b) To draw the graph a table of input and output values must be constructed first. See Table 2.

Table 2

t	-2	-1	0	1	2
$y = g(t)$	9	3	1	3	9

Each pair of t and y values in the table is plotted as a single point shown as • in Figure 8. The points are then joined with a smooth curve to produce the required graph.

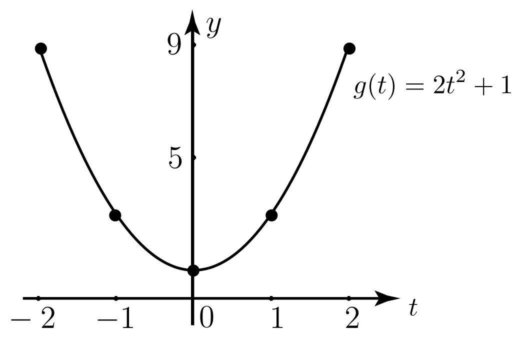


Figure 8: Graph of $g(t) = 2t^2 + 1$

(c) The range is the set of values which the function takes. By inspecting the graph we see that the range of g is the interval $1 \leq g(t) \leq 9$.

Task

Consider the function given by $f(x) = x^2 + 2, \quad -3 \leq x \leq 3$

(a) State the domain of the function:

Your solution

Recall that the domain of a function $f(x)$ is the set of values that x is allowed to take. Write down this set of values:

Answer

$-3 \leq x \leq 3$

(b) Draw up a table of input and output values for this function:

Your solution

The table of values has been partially calculated. Complete this now:

input, x	-3	-2	-1	0	1	2	3
output, $x^2 + 2$		6		2			

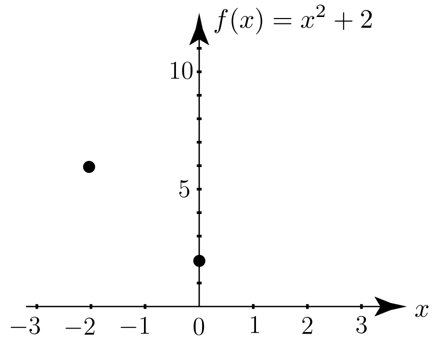
Answer

x	-3	-2	-1	0	1	2	3
$x^2 + 2$	11	6	3	2	3	6	11

(c) Plot a graph of the function:

Your solution

Part of the graph $f(x) = x^2 + 2$ is shown in the figure. Complete it.



(d) Deduce the range of the function by inspecting the graph:

Your solution

Recall that the range of the function is the set of values that the function takes as x is varied. It is possible to deduce this from the graph. Write this set as an interval.

Answer

(d) $[2, 11]$

Exercises

1. Explain the meaning of the terms 'dependent variable' and 'independent variable'. When plotting a graph, which variables are plotted on which axes ?
2. When stating the coordinates of a point, which coordinate is given first ?
3. Explain the meaning of an expression such as $y(x)$ in the context of functions. What is the interpretation of $x(t)$?
4. Explain the meaning of the terms 'domain' and 'range' when applied to functions.
5. Plot a graph of the following functions. In each case state the domain and the range of the function.
   (a) $f(x) = 3x + 2, \quad -2 \leq x \leq 5$
   (b) $g(x) = x^2 + 4, \quad -2 \leq x \leq 3$
   (c) $p(t) = 2t^2 + 8, \quad -2 \leq t \leq 4$
   (d) $f(t) = 6 - t^2, \quad 1 \leq t \leq 5$
6. Explain why the value $x = -7$ should be excluded from the domain of $f(x) = \frac{5}{x+7}$.
7. What value(s) should be excluded from the domain of $f(t) = \frac{1}{t^2}$?

Answers

1. The independent variable is plotted on the horizontal axis.
2. The independent variable is given first, as in (x, y).
3. $x(t)$ means that the dependent variable x is a function of the independent variable t.
5. (a) domain $[-2, 5]$, range $[-4, 17]$, (b) $[-2, 3], [4, 13]$, (c) $[-2, 4], [8, 40]$, (d) $[1, 5], [-19, 5]$.
6. f is undefined when $x = -7$.
7. $t = 0$.

2. Parametric representation of a function

Suppose we write x and y in terms of t in the form

$$x = 4t \qquad y = 2t^2, \qquad \text{for } -1 \leq t \leq 1 \tag{1}$$

For different values of t between -1 and 1, we can calculate pairs of values of x and y. For example when $t = 1$ we see that $x = 4(1) = 4$ and $y = 2 \times 1^2 = 2$. That is, $t = 1$ corresponds to the point with (x, y) coordinates $(4, 2)$.
A table of values is given in Table 3.

Table 3

t	-1	-0.5	0	0.5	1
x	-4	-2	0	2	4
y	2	0.5	0	0.5	2

If the resulting points are plotted on a graph then different values of t correspond to different points on the graph. The graph of (1) is plotted in Figure 9.

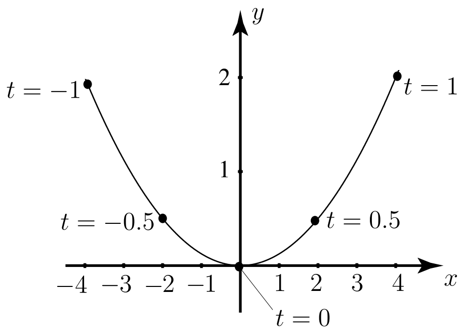


Figure 9: Graph of the function defined parametrically by $x = 4t$, $y = 2t^2$, $-1 \leq t \leq 1$

It is often possible to convert a parametric representation of a function into the more usual form by combining the two expressions to eliminate the parameter. Thus if $x = 4t$ we can write $t = \dfrac{x}{4}$ and so

$$\begin{aligned} y = 2t^2 &= 2\left(\frac{x}{4}\right)^2 \\ &= \frac{2x^2}{16} \\ &= \frac{x^2}{8} \end{aligned}$$

Using $y = \dfrac{x^2}{8}$ we can, by giving x values, find corresponding values of y. Plotting these (x, y) values gives, of course, exactly the same curve as in Figure 9.

Consider the function $x = \frac{1}{2}\left(t + \frac{1}{t}\right)$, $y = \frac{1}{2}\left(t - \frac{1}{t}\right)$, $1 \leq t \leq 8$.
(a) Draw up a table of values of this function.
(b) Plot a graph of the function

Your solution

(a) A partially completed table of values has been prepared. Complete the table.

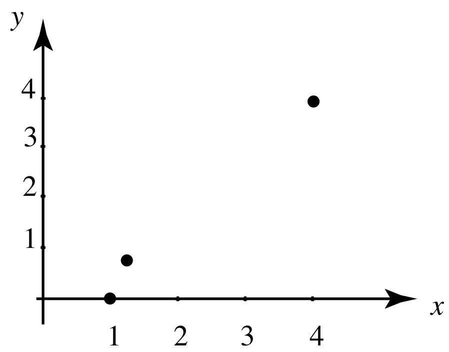
t	1	2	3	4	5	6	7	8
x	1	1.25	1.67					4.06
y	0	0.75						3.94

Answer

t	1	2	3	4	5	6	7	8
x	1	1.25	1.67	2.13	2.60	3.08	3.57	4.06
y	0	0.75	1.33	1.88	2.40	2.92	3.43	3.94

Your solution

(b) The graph is shown in the figure. Add your points to those already marked on the graph.



It is possible to eliminate t between the two equations so that the original parametric form can be expressed as $x^2 - y^2 = 1$.

Task

A particle with mass m falls under gravity so that at time t its distance from the y-axis is $2t$ and its distance from the x-axis is $-mg\dfrac{t^2}{2}+3$ where g is a constant (the acceleration due to gravity). Find the value of t when the particle crosses the x-axis and, at this time, find the distance from the y-axis.

Begin by obtaining the parametric equations of the path of the particle:

Your solution

$x =$ $\qquad\qquad$ $y =$

Answer

$x = 2t \qquad y = -mg\dfrac{t^2}{2}+3$

Now find the value of t when $y = 0$:

Your solution

$t =$

Answer

$t = \sqrt{6/(mg)}$

Finally, obtain the value of x at this value of t:

Your solution

$x =$

Answer

$x = 2\sqrt{6/(mg)}$

Exercises

1. Explain what is meant by the term 'parameter'.
2. Consider the parametric equations $x = \sqrt{t}$, $y = t$, for $t \geq 0$.
   (a) Draw up a table of values of t, x and y for values of t between 0 and 10.
   (b) Plot a graph of this function.
   (c) Obtain an explicit equation for y in terms of x.

Answers

2. (c) $y = x^2$, $0 \leq x \leq \sqrt{10}$

One-to-One and Inverse Functions

2.3

Introduction

In this Section we examine more terminology associated with functions. We explain one-to-one and many-to-one functions and show how the rule associated with certain functions can be reversed to give so-called inverse functions. These ideas will be needed when we deal with particular functions in later Sections.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a function
- be able to sketch graphs of simple functions

Learning Outcomes

On completion you should be able to ...

- explain what is meant by a one-to-one function
- explain what is meant by a many-to-one function
- explain what is meant by an inverse function, and determine when and how such a function can be found

1. One-to-many rules, many-to-one and one-to-one functions

One-to-many rules

Recall from Section 2.1 that a rule for a function must produce a single output for a given input. Not all rules satisfy this criterion. For example, the rule 'take the square root of the input' cannot be a rule for a function because for a given input there are two outputs; an input of 4 produces outputs of 2 and -2. Figure 10 shows two ways in which we can picture this situation, the first being a block diagram, and the second using two sets representing input and output values and the relationship between them.

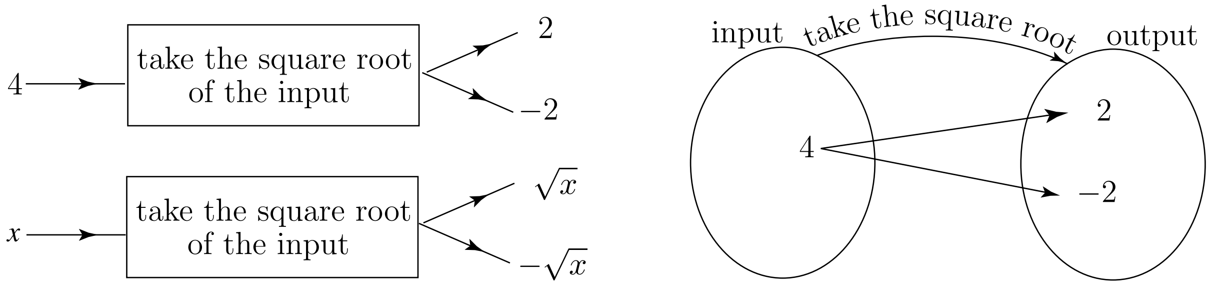


Figure 10: This rule cannot be a function - it is a one-to-many rule

Such a rule is described as a **one-to-many rule**. This means that one input produces more than one output. This is obvious from inspecting the sets in Figure 10.
The graph of the rule 'take $\pm\sqrt{x}$' can be drawn by constructing a table of values:

Table 4

x	0	1	2	3	4
$y = \pm\sqrt{x}$	0	± 1	$\pm\sqrt{2}$	$\pm\sqrt{3}$	± 2

The graph is shown in Figure 11(a). For each value of x there are two corresponding values of y. Plotting a graph of a one-to-many rule will result in a curve through which a vertical line can be drawn which cuts the curve more than once as you can see. The vertical line cuts the curve more than once because there is more than one y value for each x value.

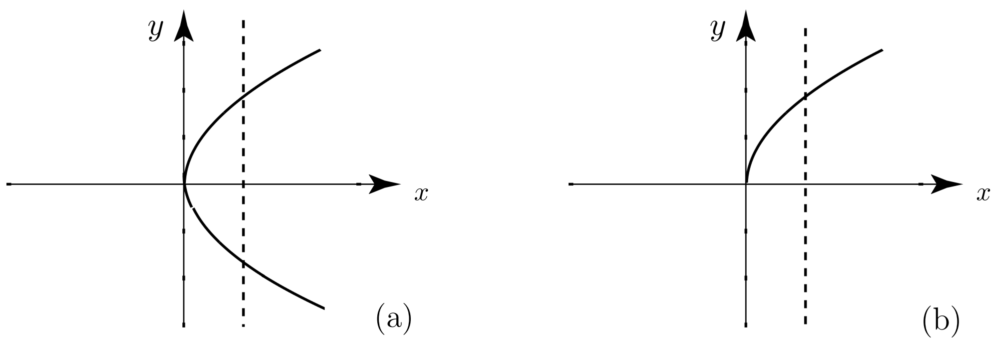


Figure 11

By describing a rule more carefully it is possible to make sure a single output results from a single input, thereby defining a valid rule for a function. For example, the rule 'take the **positive** square root of the input' is a valid function rule because a given input produces a single output. The graph of this function is displayed in Figure 11(b).

Many-to-one and one-to-one functions

Consider the function $y(x) = x^2$. An input of $x = 3$ produces an output of 9. Similarly, an input of -3 also produces an output of 9. In general, a function for which different inputs can produce the same output is called a **many-to-one function**. This is represented pictorially in Figure 12 from which it is clear why we call this a many-to-one function.

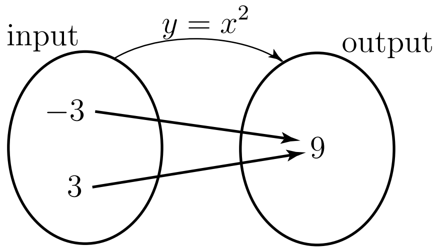


Figure 12: This represents a many-to-one function

Note that whilst this is many-to-one it is still a function since any chosen input value has only one arrow emerging from it. Thus there is a single output for each input.
It is possible to decide if a function is many-to-one by examining its graph. Consider the graph of $y = x^2$ shown in Figure 13.

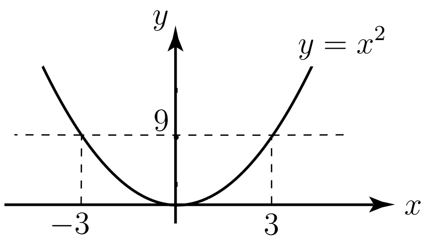


Figure 13: The function $y = x^2$ is a many-to-one function

We see that a horizontal line drawn on the graph cuts it more than once. This means that two (or more) different inputs have yielded the same output and so the function is many-to-one.
If a function is not many-to-one then it is said to be **one-to-one**. This means that each different input to the function yields a different output.

Consider the function $y(x) = x^3$ which is shown in Figure 14. A horizontal line drawn on this graph will intersect the curve only once. This means that each input value of x yields a different output value for y.

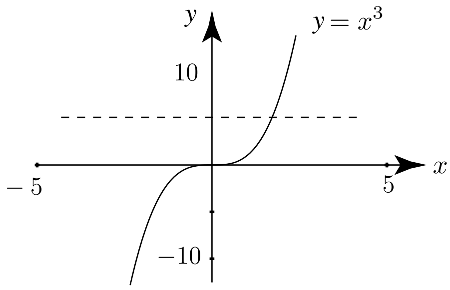


Figure 14: The function $y(x) = x^3$ is a one-to-one function

Task

Study the graphs shown in Figure 15. Decide which, if any, are graphs of functions. For those which are, state if the function is one-to-one or many-to-one.

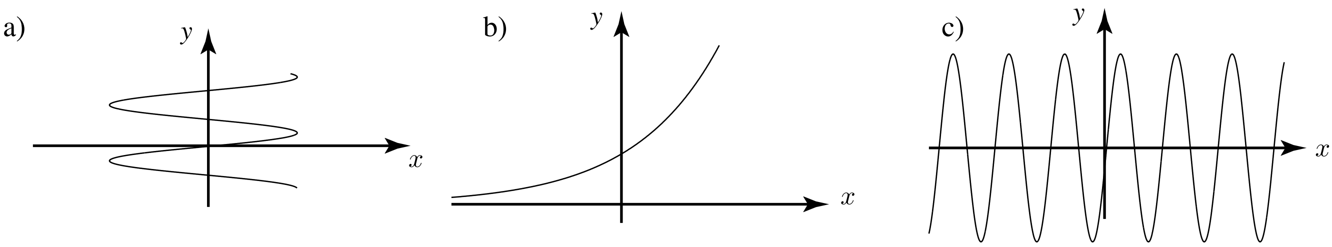


Figure 15

Your solution

Answer

(a) not a function, (b) one-to-one function, (c) many-to-one function

2. Inverse of a function

We have seen that a function can be regarded as taking an input, x, and processing it in some way to produce a single output $f(x)$ as shown in Figure 16(a). A natural question to ask is whether we can find another function that will reverse the process. In other words, can we find a function that will start with $f(x)$ and process it to produce x again? This idea is also shown in Figure 16(b). If we can find such a function it is called the **inverse function** to $f(x)$ and is given the symbol $f^{-1}(x)$. Do not confuse the '-1' with an index, or power. Here the superscript is used purely as the notation for the inverse function. Note that the composite function $f^{-1}(f(x)) = x$ as shown in Figure 17.

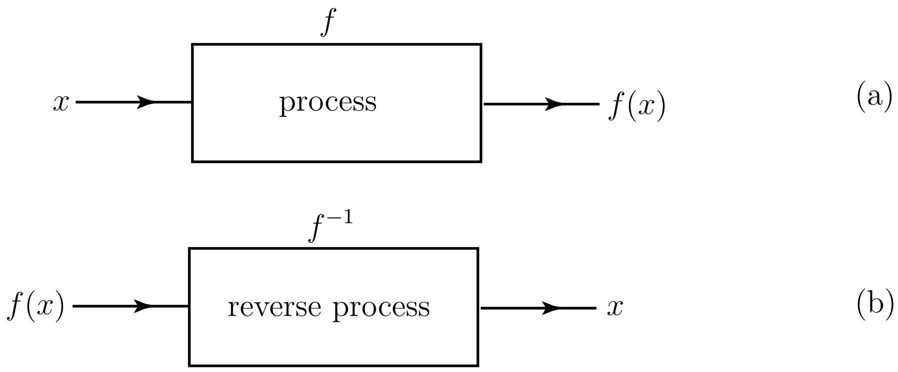


Figure 16: The second block reverse the process in the first

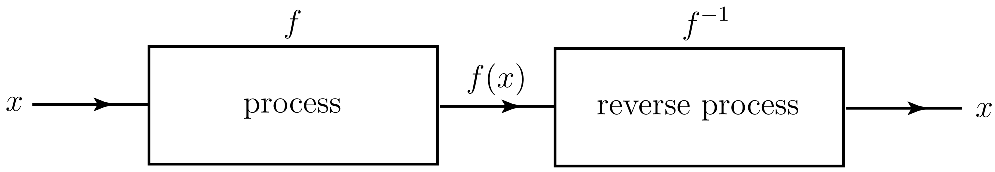


Figure 17: f^{-1} reverses the process in f

Example 6

Find the inverse function to $f(x) = 3x - 8$.

Solution

The given function takes an input, x and produces an output $3x - 8$. The inverse function, f^{-1}, must take an input $3x - 8$ and give an output x. That is

$$f^{-1}(3x - 8) = x$$

If we introduce a new variable $z = 3x - 8$, and transpose this for x to give

$$x = \frac{z+8}{3} \qquad \text{then} \qquad f^{-1}(z) = \frac{z+8}{3}$$

So the rule for f^{-1} is add 8 to the input and divide the result by 3. Writing f^{-1} with x as its argument gives

$$f^{-1}(x) = \frac{x+8}{3}$$

This is the inverse function.

Not all functions possess an inverse function. In fact, only one-to-one functions do so. If a function is many-to-one the process to reverse it would require many outputs from one input contradicting the definition of a function.

Task

Find the inverse of the function $f(x) = 7 - 3x$, using the fact that the inverse function must take an input $7-3x$ and produce an output x. So $f^{-1}(7-3x) = x$

Introduce a new variable z so that $z = 7 - 3x$ and transpose this to find x. Hence write down the inverse function:

Your solution

Answer

$f^{-1}(z) = \dfrac{7-z}{3}$. With x as its argument the inverse function is $f^{-1}(x) = \dfrac{7-x}{3}$.

Exercises

1. Explain why a one-to-many rule cannot be a function.
2. Illustrate why $y = x^4$ is a many-to-one function by providing a suitable example.
3. By sketching a graph of $y = 3x - 1$ show that this is a one-to-one function.
4. Explain why a many-to-one function does not have an inverse function. Give an example.
5. Find the inverse of each of the following functions:

   (a) $f(x) = 4x + 7$, (b) $f(x) = x$, (c) $f(x) = -23x$, (d) $f(x) = \dfrac{1}{x+1}$.

Answers

5. (a) $f^{-1}(x) = \dfrac{x-7}{4}$, (b) $f^{-1}(x) = x$, (c) $f^{-1}(x) = -\dfrac{x}{23}$, (d) $f^{-1}(x) = \dfrac{1-x}{x}$.

Characterising Functions

2.4

Introduction

There are a number of different terms used to describe the ways in which functions behave. In this Section we explain some of these terms and illustrate their use.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a function
- be able to graph simple functions

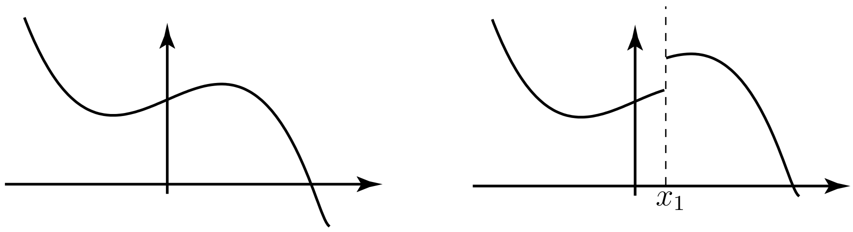
Learning Outcomes

On completion you should be able to ...

- explain the distinction between a continuous and discontinuous function
- find the limits of simple functions
- explain what is meant by a periodic function
- explain what is meant by an odd function and an even function

1. Continuous and discontinuous functions and limits

Look at the graph shown in Figure 18a. The curve can be traced out from left to right without moving the pen from the paper. The function represented by this curve is said to be **continuous** at every point. If we try to trace out the curve in Figure 18b, the presence of a jump in the graph (at $x = x_1$) means that the pen must be lifted from the paper and moved in order to trace the graph. Such a function is said to be **discontinuous** at the point where the jump occurs. The jumps are known as **discontinuities.**



(a) (b)

Figure 18: (a) A continuous function (b) A discontinuous function

Sketch a graph of a function which has two discontinuities.

Your solution

(Get your tutor to check it.)

When defining a discontinuous function algebraically it is often necessary to give different function rules for different values of x. Consider, for example, the function defined as:

$$f(x) = \begin{cases} 3 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

Notice that there is one rule for when x is less than 0 and another rule for when x is greater than or equal to 0.

A graph of this function is shown in Figure 19.

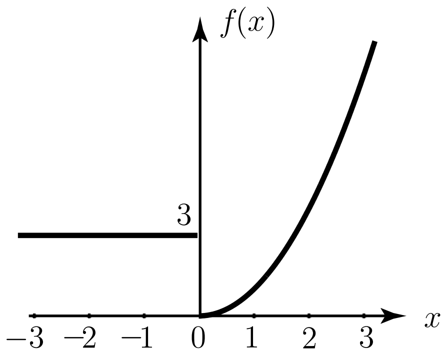


Figure 19: An example of a discontinuous function

Suppose we ask 'to what value does y approach as x approaches 0?'. From the graph we see that as x gets nearer and nearer to 0, the value of y gets nearer to 0, if we approach from the right-hand side. We write this formally as

$$\lim_{x \to 0^+} f(x) = 0$$

and say 'the limit of $f(x)$ as x tends to 0 from above is 0.'

On the other hand if x gets closer to zero, from the left-hand side, the value of y remains at 3. In this case we write

$$\lim_{x \to 0^-} f(x) = 3$$

and say 'the limit of $f(x)$ as x tends to 0 from below is 3.'

In this example the right-hand limit and the left-hand limit are not equal, and this is indicative of the fact that the function is discontinuous.

In general a function is continuous at a point $x = a$ if the left-hand and right-hand limits are the same there and are finite, and if both of these are equal to the value of the function at that point. That is

Key Point 2

A function $f(x)$ is continuous at $x = a$ if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$$

If the right-hand and left-hand limits are the same, we can simply describe this common limit as $\lim\limits_{x \to a} f(x)$. If the limits are not the same we say the limit of the function does not exist at $x = a$.

Exercises

1. Explain the distinction between a continuous and a discontinous function. Draw a graph showing an example of each type of function.
2. Study graphs of the functions $y = x^2$ and $y = -x^2$. Are these continuous functions?
3. Study graphs of $y = 3x - 2$ and $y = -7x + 1$. Are these continuous functions?
4. Draw a graph of the function

$$f(x) = \begin{cases} 2x+1 & x < 3 \\ 5 & x = 3 \\ 6 & x > 3 \end{cases}$$

Find

(a) $\lim_{x \to 0^+} f(x)$, (b) $\lim_{x \to 0^-} f(x)$, (c) $\lim_{x \to 0} f(x)$, (d) $\lim_{x \to 3^+} f(x)$, (e) $\lim_{x \to 3^-} f(x)$,

(f) $\lim_{x \to 3} f(x)$,

Answers 2. Yes. 3. Yes. 4. (a) 1, (b) 1, (c) 1, (d) 6, (e) 7, (f) limit does not exist.

2. Periodic functions

Any function that has a definite pattern repeated at regular intervals is said to be periodic. The interval over which the repetition takes place is called the **period** of the function, and is usually given the symbol T. The period of a periodic function is usually obvious from its graph.
Figure 20 figure shows a graph of a periodic function with period $T = 3$. This function has discontinuities at values of x which are divisible by 3.

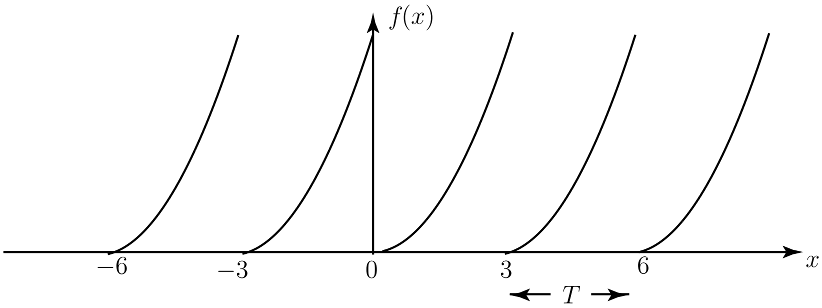


Figure 20

Figure 21 shows a graph of a periodic function with period $T = 6$. This function has no discontinuities.

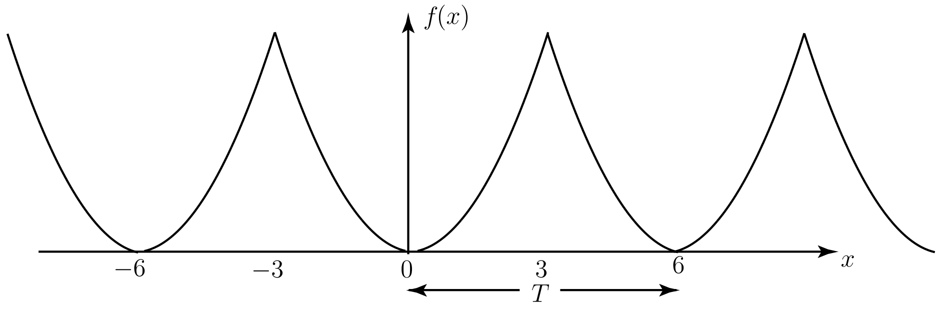


Figure 21

If a function is a periodic function with period T then, for any value of the independent variable x, the value of $f(x + T)$ is the **same** as the value of $f(x)$.

Key Point 3

A function $f(x)$ is **periodic** if we can find a number T such that

$$f(x + T) = f(x) \qquad \text{for all values of } x$$

Often a periodic function will be defined by simply specifying the period of the function and by stating the rule for the function within one period. This information alone is sufficient to draw the graph for all values of the independent variable.

Figure 22 shows a graph of the periodic function defined by

$$f(x) = x, \quad -\pi < x < \pi, \qquad \text{period } T = 2\pi$$

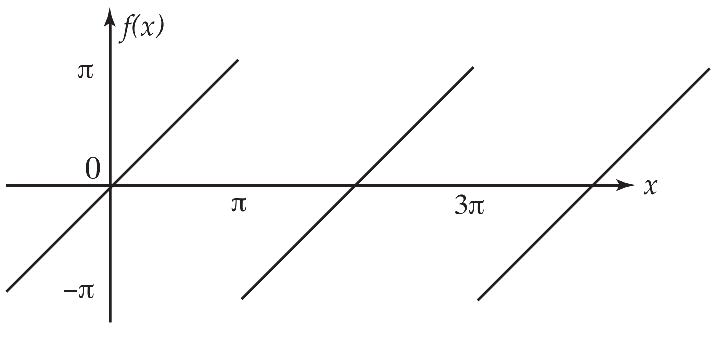


Figure 22

Exercises

1. Explain what is meant by a periodic function.
2. Sketch a graph of a periodic function which has no discontinuities.
3. Sketch a graph of a periodic function which has discontinuities.
4. A periodic function has period 0.01 seconds. How many times will the pattern in the graph repeat over an interval of 10 seconds ?

Answer 4. 1000.

3. Odd and even functions

Example 7

Figure 23 shows graphs of several functions. They share a common property. Study the graphs and comment on any symmetry.

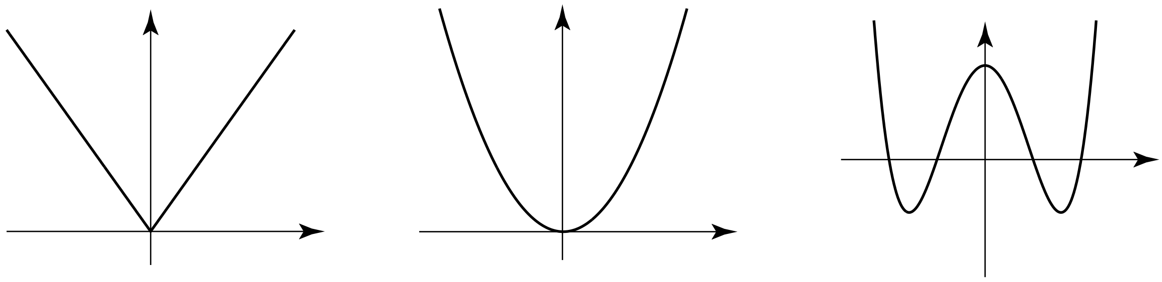


Figure 23

Solution

The graphs are all symmetrical about the y axis.

Any function which is symmetrical about the y axis, i.e. where the graph of the right-hand part is the mirror image of that on the left, is said to be an **even function**. Even functions have the following property:

Key Point 4

Even Function

An even function is such that $f(-x) = f(x)$ for all values of x.

Key Point 4 is saying that the function value at a negative value of x is the same as the function value at the corresponding positive value of x.

Example 8

Show algebraically that $f(x) = x^4 + 5$ is an even function.

Solution

We must show that $f(-x) = f(x)$.

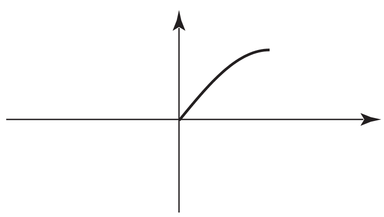
$$f(-x) = (-x)^4 + 5 = x^4 + 5$$

Hence $f(-x) = f(x)$ and so the function is even. Check for yourself that $f(-3) = f(3)$.

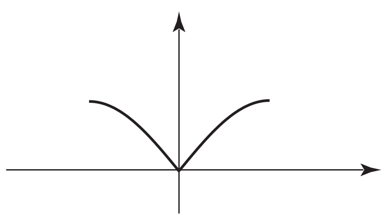
Task

Extend the graph in the solution box in order to produce a graph of an even function.

Your solution

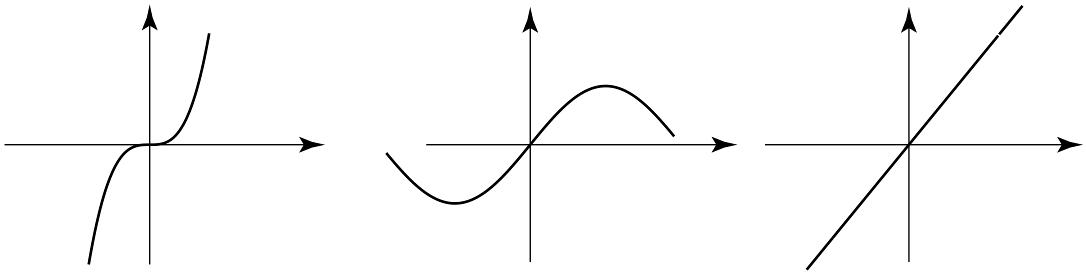


Answer



Task

The following diagrams shows graphs of several functions. They share a common property. Study the graphs and comment on any symmetry.



Your solution

Answer

There is rotational symmetry about the origin. That is, each curve, when rotated through $180°$, transforms into itself.

Any function which possesses such symmetry – that is the graph of the right can be obtained by rotating the curve on the left through $180°$ about the origin – is said to be an **odd** function. Odd functions have the following property:

Key Point 5

Odd Function

An odd function is such that $f(-x) = -f(x)$ for all values of x.

Key Point 5 is saying that the function value at a negative value of x is minus the function value at the corresponding positive value of x.

Example 9

Show that the function $f(x) = x^3 + 4x$ is odd.

Solution

We must show that $f(-x) = -f(x)$.

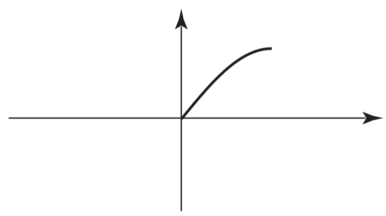
$$\begin{aligned} f(-x) &= (-x)^3 + 4(-x) \\ &= -x^3 - 4x \\ &= -(x^3 + 4x) \\ &= -f(x) \end{aligned}$$

and so this function is odd. Check for yourself that $f(-2) = -f(2)$.

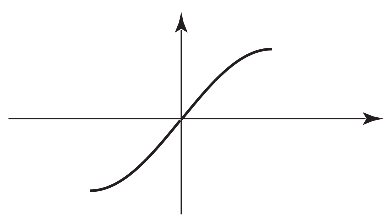
Task

Extend the graph in the solution box in order to produce a graph of an odd function.

Your solution



Answer



Note that some functions are **neither odd nor even**; for example $f(x) = x^3 + x^2$ is neither even nor odd.

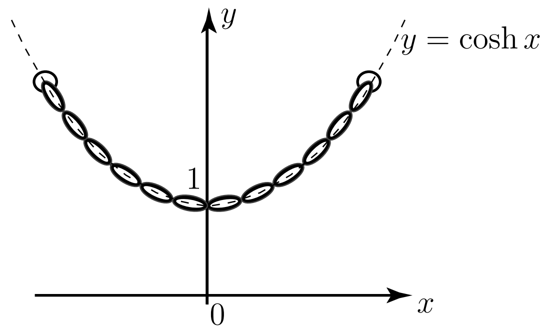
The reader should confirm (with simple examples) that, 'odd' and 'even' functions have the following properties:

odd + odd = odd	even + even = even	odd + even = neither
odd × odd = even	even × even = even	odd × even = odd

Exercises

1. Classify the following functions as odd, even or neither. If necessary sketch a graph to help you decide. (a) $f(x) = 6$, (b) $f(x) = x^2$, (c) $f(x) = 2x + 1$, (d) $f(x) = x$, (e) $f(x) = 2x$

2. The diagram below represents a heavy cable hanging under gravity from two points at the same height. Such a curve (shown as a dashed line), known as a **catenary**, is described by a mathematical function known as a hyperbolic cosine, $f(x) = \cosh x$, discussed in HELM 6.



A catenary

(a) Comment upon any symmetry.

(b) Is this function one-to-one or many-to-one?

(c) Is this a continuous or discontinuous function?

(d) State $\lim_{x\to 0} \cosh x$.

Answers

1(a) even, (b) even, (c) neither, (d) odd, (e) odd

2(a) function is even, symmetric about the y-axis, (b) many-to-one, (c) continuous, (d) 1

The Straight Line 2.5

Introduction

Probably the most important function and graph that you will use are those associated with the straight line. A large number of relationships between engineering variables can be described using a straight line or **linear** graph. Even when this is not strictly the case it is often possible to approximate a relationship by a straight line. In this Section we study the equation of a straight line, its properties and graph.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a function
- be able to graph simple functions

Learning Outcomes

On completion you should be able to ...

- recognise the equation of a straight line
- explain the significance of a and b in the equation of a line $f(x) = ax + b$
- find the gradient of a straight line given two points on the line
- find the equation of a straight line through two points
- find the distance between two points

1. Linear functions

Any function of the form $y = f(x) = ax + b$ where a and b are constants is called a **linear function.** The constant a is called the **coefficient of** x, and b is referred to as the **constant term.**

Key Point 6

All linear functions can be written in the form:

$$f(x) = ax + b$$

where a and b are constants.

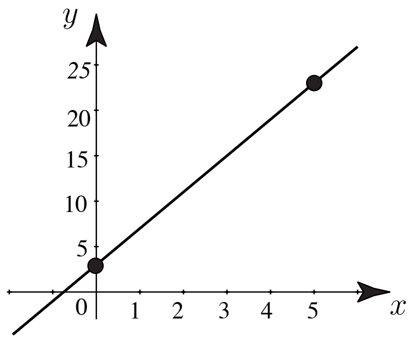
For example, $f(x) = 3x + 2$, $g(x) = \frac{1}{2}x - 7$, $h(x) = -3x + \frac{2}{3}$ and $k(x) = 2x$ are all linear functions. The graph of a linear function is always a straight line. Such a graph can be plotted by finding just two distinct points and joining these with a straight line.

Example 10

Plot the graph of the linear function $y = f(x) = 4x + 3$.

Solution

We start by finding two points. For example if we choose $x = 0$, then $y = f(0) = 3$, i.e. the first point has coordinates $(0, 3)$. Secondly, suppose we choose $x = 5$, then $y = f(5) = 23$. The second point is $(5, 23)$. These two points are then plotted and then joined by a straight line as shown in the following diagram.



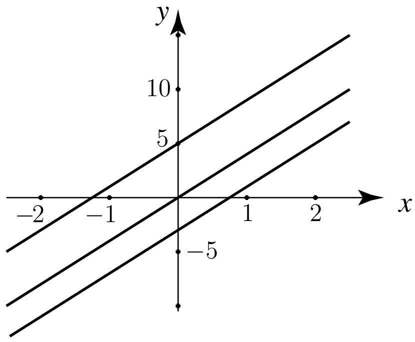
Example 11

Plot graphs of the three linear functions $y = 4x - 3$, $y = 4x$, and $y = 4x + 5$, for $-2 \leq x \leq 2$.

Solution

For each function it is necessary to find two points on the line.

For $y = 4x - 3$, suppose for the first point we choose $x = 0$, so that $y = -3$. For the second point, let $x = 2$ so that $y = 5$. So, the points $(0, -3)$ and $(2, 5)$ can be plotted and joined. This is shown in the following diagram.



For $y = 4x$ we find the points $(0, 0)$ and $(2, 8)$. Similarly for $y = 4x + 5$ we find points $(0, 5)$ and $(2, 13)$. The corresponding lines are also shown in the figure.

Task

Refer to Example 11. Comment upon the effect of changing the value of the constant term of the linear function.

Your solution

Answer

As the constant term is varied, the line moves up or down the page always remaining parallel to its initial position.

The value of the constant term is also known as the **vertical** or ***y*-axis intercept** because this is the value of y where the line cuts the y axis.

Task

State the vertical intercept of each of the following lines:
(a) $y = 3x + 3$, (b) $y = \frac{1}{2}x - \frac{1}{3}$, (c) $y = 1 - 3x$, (d) $y = -5x$.

In each case you need to identify the constant term:

Your solution

(a) (b) (c) (d)

Answer

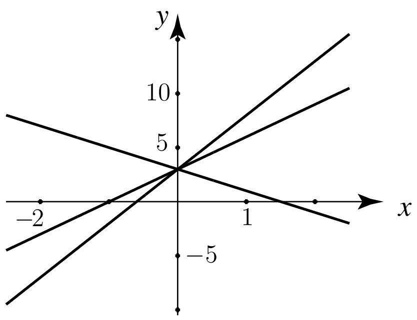
(a) 3, (b) $-\frac{1}{3}$, (c) 1, (d) 0

Example 12

Plot graphs of the lines $y = 3x + 3$, $y = 5x + 3$ and $y = -2x + 3$.

Solution

Note that all three lines have the same constant term, that is 3. So all three lines pass through $(0, 3)$, the vertical intercept. A further point has been calculated for each of the lines and their graphs are shown in the following diagram.



Note from the graphs in Example 12 that as the coefficient of x is changed the gradient of the graph changes. The coefficient of x gives the **gradient** or **slope** of the line. In general, for the line $y = ax + b$ a positive value of a produces a graph which slopes upwards from left to right. A negative value of a produces a graph which slopes downwards from left to right. If a is zero the line is horizontal, that is its gradient is zero. These properties are summarised in the next figure.

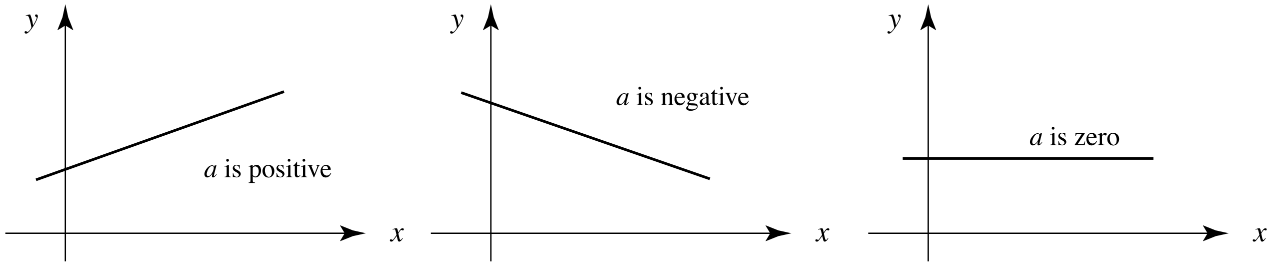


Figure 24: The gradient of a line $y = ax + b$ depends upon the value of a.

Key Point 7

Linear Equation

In the linear function $f(x) = ax + b$, a is the gradient and b is the vertical intercept.

Task

State the gradients of the following lines:

(a) $y = 7x + 2$ (b) $y = -\dfrac{1}{3}x + 4$ (c) $y = \dfrac{x+2}{3}$

In each case the coefficient of x must be examined:

Your solution

(a) (b) (c)

Answer

(a) 7, (b) $-1/3$, (c) $1/3$

Task

Which of the following lines has the steepest gradient ?
(a) $y = \dfrac{17x+4}{5}$, (b) $y = 9x - 2$, (c) $y = \dfrac{1}{3}x + 4$.

Your solution

Answer

(b) because the three gradients are (a) $\dfrac{17}{5}$ (b) 9 (c) $\dfrac{1}{3}$

Exercises

1. State the general form of the equation of a straight line explaining the role of each of the terms in your answer.
2. State which of the following functions will have straight line graphs.

   (a) $f(x) = 3x - 3$, (b) $f(x) = x^{1/2}$, (c) $f(x) = \dfrac{1}{x}$, (d) $f(x) = 13$, (e) $f(x) = -2 - x$.
3. For each of the following, identify the gradient and vertical intercept.

   (a) $f(x) = 2x + 1$, (b) $f(x) = 3$, (c) $f(x) = -2x$, (d) $f(x) = -7 - 17x$,

   (e) $f(x) = mx + c$.

Answers

1. e.g. $y = ax + b$. x is the independent variable, y is the dependent variable, a is the gradient and b is the vertical intercept.
2. (a), (d) and (e) will have straight line graphs.
3. (a) gradient = 2, vertical intercept =1, (b) $0, 3$, (c) $-2, 0$, (d) $-17, -7$, (e) m, c.

2. The gradient of a straight line through two points

A common requirement is to find the gradient of a line when we know the coordinates of two points on it. Suppose the two points are $A(x_1, y_1)$, $B(x_2, y_2)$ as shown in the following figure.

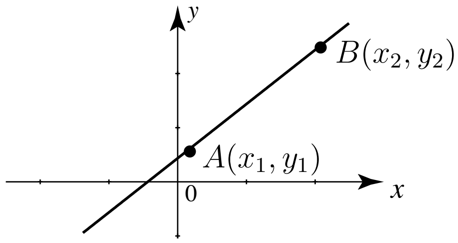


Figure 25

The gradient of the line joining A and B can be calculated from the following formula.

Key Point 8

Gradient of Line Through Two Points

The gradient of the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

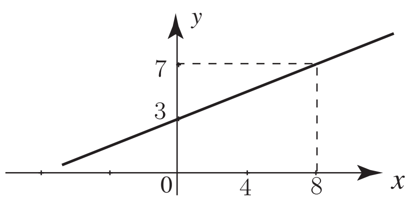
Example 13

Find the gradient of the line joining the points $A(0, 3)$ and $B(4, 5)$.

Solution

We calculate the gradient as follows:

$$\begin{aligned} \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5-3}{4-0} \\ &= \frac{1}{2} \end{aligned}$$



Thus the gradient of the line is $\frac{1}{2}$. Graphically, this means that when x increases by 1, the value of y increases by $\frac{1}{2}$.

Task

Find the gradient of the line joining the points $A(-1, 4)$ and $B(2, 1)$.

Your solution

gradient $= \dfrac{y_2 - y_1}{x_2 - x_1} =$

Answer

$$\frac{1-4}{2-(-1)} = -1$$

Thus the gradient of the line is -1. Graphically, this means that when x increases by 1, the value of y decreases by 1.

Exercises

1. Calculate the gradient of the line joining $(1, 0)$ and $(15, -3)$.
2. Calculate the gradient of the line joining $(10, -3)$ and $(15, -3)$.

Answers

1. $-3/14$. 2. 0

3. The equation of a straight line through two points

The equation of the line passing through the points with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by the following formula.

Key Point 9

The line passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \qquad \text{or, equivalently} \qquad y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Task

Find the equation of the line passing through $A(-7, 11)$ and $B(1, 3)$.

First apply the formula: $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

Your solution

$$\frac{y - \quad}{\qquad} = \frac{\qquad}{\qquad}$$

Answer

$$\frac{y - 11}{3 - 11} = \frac{x + 7}{1 + 7}.$$

Simplify this to obtain the required equation:

Your solution

Answer

$y = 4 - x$

Exercises

1. Find the equation of the line joining $(1, 5)$ and $(-9, 2)$.
2. Find the gradient and vertical intercept of the line joining $(8, 1)$ and $(-2, -3)$.

Answers 1. $y = \dfrac{3}{10}x + \dfrac{47}{10}$. 2. 0.4, -2.2.

4. The distance between two points

Referring again to the figure of HELM 2, the distance between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given using Pythagoras' theorem by the following formula.

Key Point 10

Distance Between Two Points

The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Task

Find the distance between $A(-7, 11)$ and $B(1, 3)$, using Key Point 10.

Your solution

Answer

$\sqrt{(1-(-7))^2 + (3-11)^2} = \sqrt{64+64} = \sqrt{128}$

Exercises

1. Find the distance between the points $(4, 5)$ and $(-17, 1)$.
2. Find the distance between the points $(-4, -5)$ and $(1, 7)$.

Answers

1. $\sqrt{457}$
2. 13

The Circle

2.6

Introduction

A circle is one of the most familiar geometrical figures and has been around a long time! In this brief Section we discuss the basic coordinate geometry of a circle - in particular the basic equation representing a circle in terms of its centre and radius.

Prerequisites

Before starting this Section you should . . .

- understand what is meant by a function and be able to use functional notation
- be able to plot graphs of functions

Learning Outcomes

On completion you should be able to . . .

- obtain the equation of any given circle
- obtain the centre and radius of a circle from its equation

1. Equations for circles in the Oxy plane

The obvious characteristic of a circle is that every point on its circumference is the **same** distance from the **centre**. This fixed distance is called the **radius** of the circle and is generally denoted by R or r or a.

In coordinate geometry terms suppose (x, y) denotes the coordinates of a point. For example, (4,2) means $x = 4,\ y = 2$, $(-1, 1)$ means $x = -1,\ y = 1$ and so on. See Figure 26.

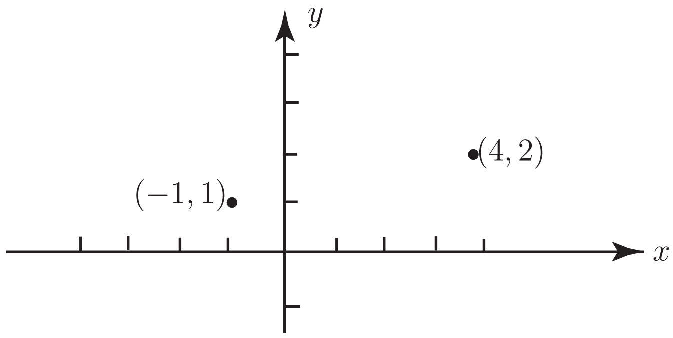


Figure 26

Example 14

Write down the distances d_1 and d_2 from the origin of the points with coordinates (4,2) and $(-1, 1)$ respectively. Generalise the result to obtain the distance d from the origin of **any** arbitrary point with coordinates (x, y).

Solution

Using Pythagoras' Theorem:

$d_1 = \sqrt{4^2 + 2^2} = \sqrt{20}$ is the distance between the origin (0,0) and the point (4,2).

$d_2 = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ is the distance between the origin and $(-1, 1)$.

$d = \sqrt{x^2 + y^2}$ is the distance from the origin to an arbitrary point (x, y). Note that the positive square root is taken in each case.

Circles with centre at the origin

Suppose (x, y) is any point P on a circle of radius R whose centre is at the origin. See Figure 27.

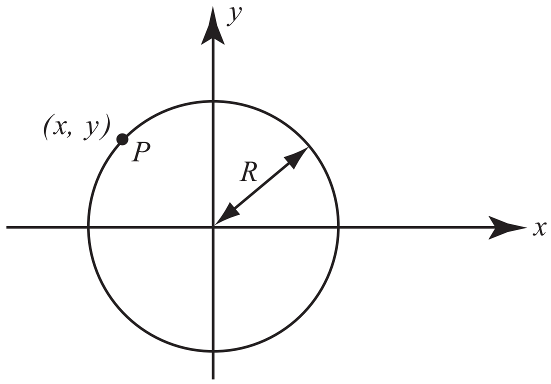


Figure 27

Using the final result of Example (14), write down an equation relating $x,\ y$ and R.

Your solution

Answer
Since $\sqrt{x^2+y^2}$ is distance of any point (x, y) from the origin, then for any point P on the above circle.

$$\sqrt{x^2+y^2} = R \qquad \text{or} \qquad x^2+y^2 = R^2$$

As the point P in Figure 27 moves around the circle its x and y coordinates change. However P will remain at the same distance R from the origin by the very definition of a circle.
Hence we say that

$$\sqrt{x^2+y^2} = R \quad \text{or, more usually,}$$

$$x^2+y^2 = R^2 \qquad (1)$$

is **the equation** of the circle radius R centre at the origin. What this means is that if a point (x, y) satisfies (1) then it lies on the circumference of the circle radius R. If (x, y) does not satisfy (1) then it does not lie on that circumference.
Note carefully that the right-hand sides of the circle equation (1) is the **square** of the radius.

Task

Consider the circle centre at the origin and of radius 5.

(a) Write down the equation of this circle.

(b) For the following points determine which lie on the circumference of this circle, which lie inside the circle and which lie outside the circle.

$(5, 0)$ $(0, -5)$ $(4, 3)$ $(-3, 4)$ $(2, \sqrt{21})$ $(-2\sqrt{6}, 1)$ $(1, 4)$ $(4, -4)$

Your solution

(a)

(b)

(x, y)	$x^2 + y^2$	conclusion
$(5, 0)$		
$(0, -5)$		
$(4, 3)$		
$(-3, -4)$		
$(2, \sqrt{21})$		
$(-2\sqrt{6}, 1)$		
$(1, 4)$		
$(4, -4)$		

Answer

(a) $x^2 + y^2 = 5^2 = 25$ is the equation of the circle.

(b) For each point (x, y) we calculate $x^2 + y^2$. If this equals 25 the point lies on the circle' if greater than 25 then outside and if less than 25 then inside.

x, y	$x^2 + y^2$	conclusion
$(5, 0)$	25	on circle
$(0, -5)$	25	on circle
$(4, 3)$	25	on circle
$(-3, -4)$	25	on circle
$(2, \sqrt{21})$	25	on circle
$(-2\sqrt{6}, 1)$	25	on circle
$(1, 4)$	17	inside circle
$(4, -4)$	32	outside circle

Figure 28 demonstrates some of the results of the previous Task.

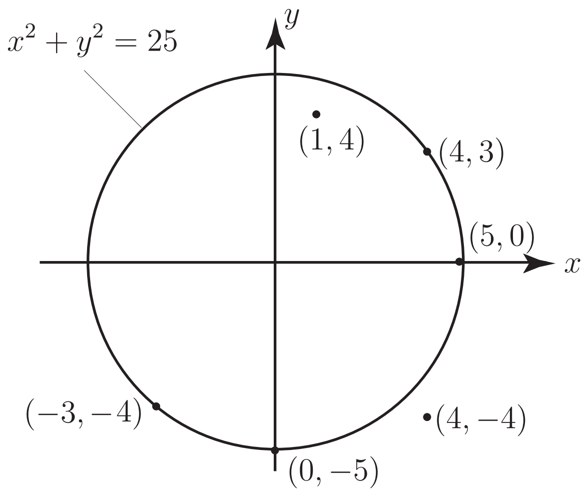
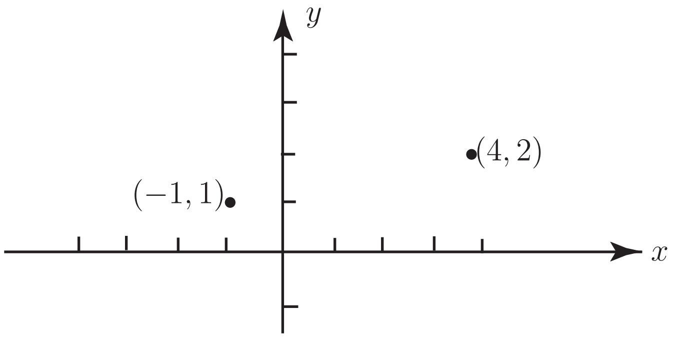


Figure 28

Note that the circle centre at the origin and of radius 1 has a special name – the **unit circle.**

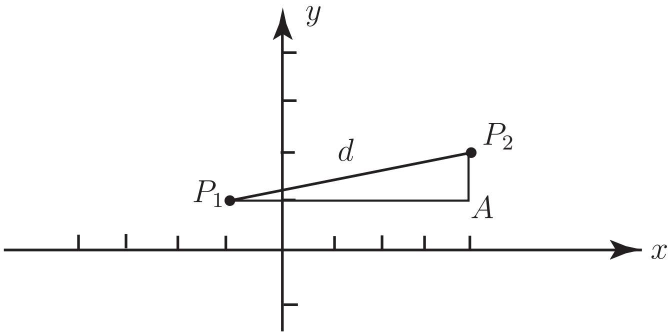
Task

Calculate the distance between the points $P_1(-1, 1)$ and $P_2(4, 2)$.



Your solution

Answer



Using Pythagoras' Theorem the distance between the two given points is

$$d = \sqrt{(P_1A)^2 + (AP_2)^2}$$

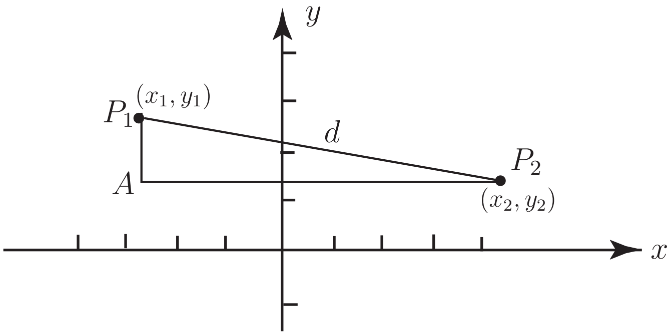
where $P_1A = 4 - (-1) = 5, \quad AP_2 = 2 - 1 = 1$

$$\therefore \quad d = \sqrt{5^2 + 1^2} = \sqrt{26}$$

Generalise your result to the previous Task to obtain the distance between any two points whose coordinates are (x_1, y_1) and (x_2, y_2).

Your solution

Answer



Between the arbitrary points P_1 and P_2 the distance is

$$d = \sqrt{(AP_2)^2 + (P_1A)^2}$$

where $AP_2 = x_2 - x_1$, $P_1A = y_1 - y_2 = -(y_2 - y_1)$

so $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Using the result of the last Task, we now consider a circle centre at the point $C(x_0; y_0)$ and of radius R. Suppose P is an arbitrary point on this circle which has co-ordinates (x, y):

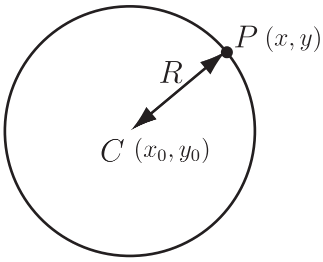


Figure 29

Clearly $R = CP = \sqrt{(x - x_0)^2 + (y - y_0)^2}$
Hence, squaring both sides,

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \qquad (2)$$

which is said to be the equation of the circle centre (x_0, y_0) radius R.

Note that if $x_0 = y_0 = 0$ (i.e. circle centre is at origin) then (2) reduces to (1) so the latter is simply a special case.

The interpretation of (2) is similar to that of (1): any point (x, y) satisfying (2) lies on the circumference of the circle.

Example 15

What does the equation $(x-3)^2+(y-4)^2=4$ represent?

Solution

It represents a circle of radius 2 (the positive square root of 4) and has centre $C\ (3,4)$.

N.B. There is no need to expand the terms on the left-hand side of the equation here. The given form reveals quite plainly the radius ($\sqrt{4}$) and centre $(3,4)$ of the circle.

Task

Write down the equations of each of the following circles for which the centre C and radius R are given:

(a) $C(0,2),\ R=2$

(b) $C(-2,0),\ R=3$

(c) $C(-3,4),\ R=5$

(d) $C(1,1),\ R=\sqrt{3}$

Your solution

(a)

(b)

(c)

(d)

Answer

(a) $x_0=0,\ y_0=2,\ R^2=4$ so by Equation (2) the circle's equation is

$$x^2+(y-2)^2=4$$

(b) $x_0=-2,\ y_0=0,\ R^2=9 \quad \therefore \quad (x+2)^2+y^2=9$

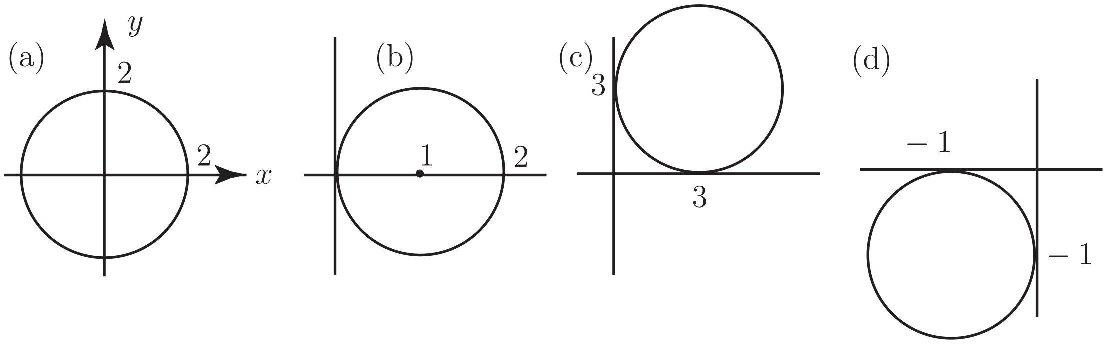
(c) $(x+3)^2+(y-4)^2=25$

(d) $(x-1)^2+(y-1)^2=3$

Again we emphasise that the right-hand side of each of these equations is the square of the radius.

Task

Write down the equations of each of the circles shown below:



Your solution

Answer

(a) $x^2 + y^2 = 4$ (centre $(0, 0)$ i.e. the origin, radius 2)

(b) $(x - 1)^2 + y^2 = 1$ (centre $(1, 0)$, radius 1)

(c) $(x - 3)^2 + (y - 3)^2 = 0$ (centre $(3, 3)$, radius 3)

(d) $(x + 1)^2 + (y + 1)^2 = 1$ (centre $(-1, -1)$, radius 1)

Consider again the equation of the circle, centre (3,4) of radius 2:

$$(x - 3)^2 + (y - 4)^2 = 4 \qquad (3)$$

In this form of the equation the centre and radius of the circle can be clearly identified and, as we said, there is no advantage in squaring out. However, if we did square out the equation would become

$$x^2 - 6x + 9 + y^2 - 8x + 16 = 4 \quad \text{or} \quad x^2 - 6x + y^2 - 8x + 21 = 0 \qquad (4)$$

Equation (4) is of course a valid equation for this circle but, we cannot immediately obtain the centre and radius from it.

Task

For the case of the general circle of radius R

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

expand out the square terms and simplify.

Your solution

Answer
We obtain

$$x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 - R^2 = 0$$

or

$$x^2 + y^2 - 2x_0x - 2y_0y + c = 0$$

where the constant $c = x_0^2 + y_0^2 - R^2$.

It follows from the above task that any equation of the form

$$x^2 + y^2 - 2gx - 2fy + c = 0 \qquad (5)$$

represents a circle with centre (g, f) and a radius obtained by solving

$$c = g^2 + f^2 - R^2$$

for R.
Thus

$$R = \sqrt{g^2 + f^2 - c} \qquad (6)$$

There is no need to remember Equation (6). In any specific problem the technique of **completion of square** can be used to turn an equation of the form (5) into the form of Equation (2) (i.e. $(x - x_0)^2 + (y - y_0)^2 = R^2$) and hence obtain the centre and radius of the circle.

NB. The key point about Equation (5) is that the coefficients of the term x^2 and y^2 are the **same**, i.e. 1. An equation with the coefficient of x^2 and y^2 identical with value $k \neq 1$ could be converted into the form (5) by division of the whole equation by k.

Task

If

$$x^2 + y^2 - 2x + 10y + 16 = 0$$

obtain the centre and radius of the circle that this equation represents.

Begin by completing the square separately on the $x-$terms and the $y-$terms:

Your solution

Answer

$$\begin{aligned} x^2 - 2x &= (x-1)^2 - 1 \\ y^2 + 10y &= (y+5)^2 - 25 \end{aligned}$$

Now complete the problem:

Your solution

Answer

The original equation

$$x^2 + y^2 - 2x + 10y + 16 = 0$$

becomes

$$(x-1)^2 - 1 + (y+5)^2 - 25 + 16 = 0$$

$$\therefore \qquad (x-1)^2 + (y+5)^2 = 10$$

which represents a circle with centre $(1, -5)$ and radius $\sqrt{10}$.

Circles and functions

Let us return to the equation of the unit circle

$$x^2 + y^2 = 1$$

Solving for y we obtain

$$y = \pm\sqrt{1 - x^2}.$$

This equation does not represent a function because of the two possible square roots which imply that for any value of x there are **two** values of y. (You will recall from earlier in this Workbook that a function requires only **one** value of the dependent variable y corresponding to each value of the independent variable x.)
However two functions can be obtained in this case: $y = y_1 = +\sqrt{1 - x^2}$ $\quad y = y_2 = -\sqrt{1 - x^2}$

whose graphs are the semicircles shown.

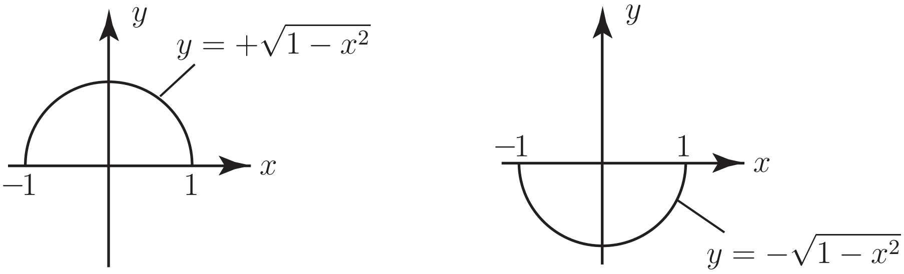


Figure 30

2. Annuli between circles

Equations in x and y, such as (1) i.e. $x^2 + y^2 = R^2$ and (2) i.e. $(x - x_0)^2 + (y - y_0)^2 = R^2$ for circles, define **curves** in the Oxy plane. However, inequalities are necessary to define **regions**. For example, the inequality

$$x^2 + y^2 < 1$$

is satisfied by all points **inside** the unit circle - for example $(0, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{4}, 0)$, $(\frac{1}{2}, \frac{1}{2})$.
Similarly $x^2 + y^2 > 1$ is satisfied by all points outside that circle such as $(1, 1)$.

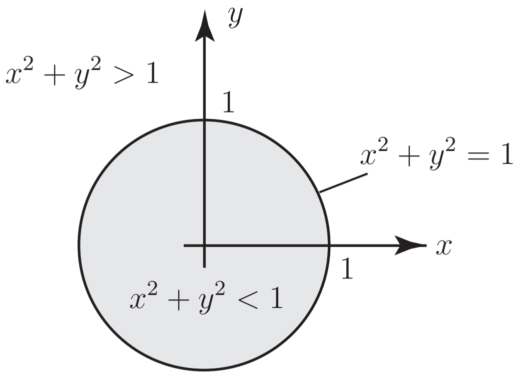


Figure 31

Example 16

Sketch the regions in the Oxy plane defined by

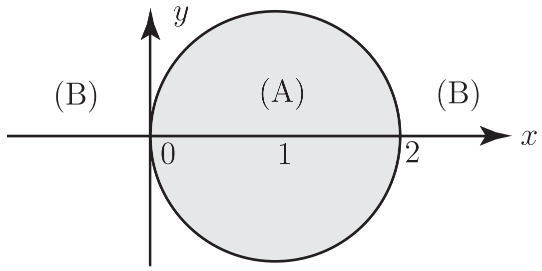
(a) $(x-1)^2 + y^2 < 1$ (b) $(x-1)^2 + y^2 > 1$

Solution

The equality $(x-1)^2+y^2 = 1$ is satisfied by any point on the circumference of the circle centre (1,0) radius 1. Then, remembering that $(x-1)^2 + y^2$ is the square of the distance between any point (x, y) and (1,0), it follows that

(a) $(x-1)^2 + y^2 < 1$ is satisfied by any point inside this circle (region (A) in the diagram.)

(b) $(x-1)^2+y^2 > 1$ defines the region exterior to the circle since this inequality is satisfied by every point outside. (Region (B) on the diagram.)



The region between two circles with the same centre (i.e. **concentric** circles) is called an **annulus** or **annular region**. An annulus is defined by **two** inequalities. For example the inequality

$$x^2 + y^2 > 1 \qquad (7)$$

defines, as we saw, the region outside the unit circle.
The inequality

$$x^2 + y^2 < 4 \qquad (8)$$

defines the region inside the circle centre origin radius 2.
Hence points (x, y) which satisfy **both** the inequalities (7) and (8) lie in the annulus between the two circles. The inequalities (7) and (8) are combined by writing

$$1 < x^2 + y^2 < 4$$

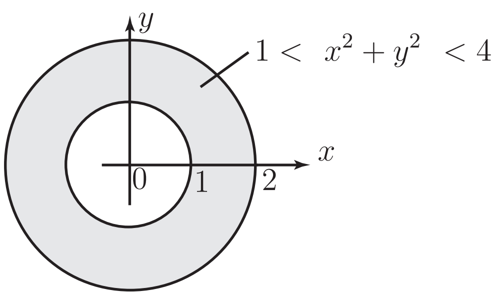


Figure 32

Task

Sketch the annulus defined by the inequalities

$$1 < (x-1)^2 + y^2 < 9$$

Your solution

Answer

The quantity $(x-1)^2 + y^2$ is the square of the distance of a point (x, y) from the point (1,0). Hence, as we saw earlier, the left-hand inequality

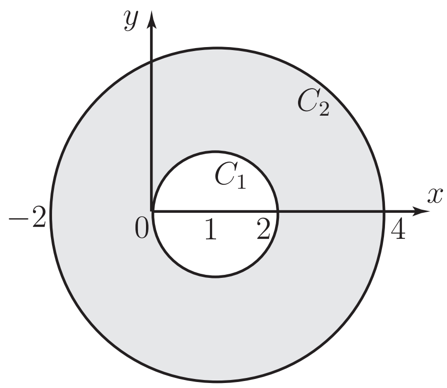
$1 < (x-1)^2 + y^2$ which is the same as $(x-1)^2 + y^2 > 1$

is the region exterior to the circle C_1 centre $(1, 0)$ radius 1.

Similarly the right-hand inequality

$$(x-1)^2 + y^2 < 9$$

defines the interior of the circle C_2 centre $(1, 0)$ radius 3. Hence the double inequality holds for any point in the annulus between C_1 and C_2.



Exercises

1. Write down the radius and the coordinates of the centre of the circle for each of the following equations

   (a) $x^2 + y^2 = 16$

   (b) $(x - 4)^2 + (y - 3)^2 = 12$

   (c) $(x + 3)^2 + (y - 1)^2 = 25$

   (d) $x^2 + (y + 1)^2 - 4 = 0$

   (e) $(x + 6)^2 + y^2 - 36 = 0$

2. Obtain in each case the equation of the given circle

   (a) centre C $(0, 0)$ radius 7

   (b) centre C $(0, 2)$ radius 2

   (c) centre C $(4, -4)$ radius 4

   (d) centre C $(-2, -2)$ radius 4

   (e) centre C $(-6, 0)$ radius 5

3. Obtain the radius and the coordinates of the centre for each of the following circles

   (a) $x^2 + y^2 - 10x + 12y = 0$

   (b) $x^2 + y^2 + 2x - 4y = 11$

   (c) $x^2 + y^2 - 6x - 16 = 0$

4. Describe the regions defined by each of these inequalities

   (a) $x^2 + y^2 > 4$

   (b) $x^2 + y^2 < 16$

   (c) the inequalities in (i) and (ii) together

5. State an inequality that describes the points that lie outside the circle of radius 4 with centre $(-4, 2)$.

6. State an inequality that describes the points that lie inside the circle of radius $\sqrt{6}$ with centre $(-2, -1)$.

7. Obtain the equation of the circle which has centre $(3, 4)$ and which passes through the point $(0, 5)$.

8. Show that if $A(x_1, y_1)$ and $B(x_2, y_2)$ are at opposite ends of a diameter of a circle then the equation of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

   (Hint: if P is any point on the circle obtain the slopes of the lines AP and BP and recall that the angle in a semicircle must be a right-angle.)

9. State the equation of the unique circle which **touches** the $x-$axis at the point (2,0) and which passes through the point $(-1, 9)$.

Answers

1. (a) radius 4 centre $(0,0)$
   (b) radius $\sqrt{12}$ centre $(4,3)$
   (c) radius 5 centre $(-3,1)$
   (d) radius 2 centre $(0,-1)$
   (e) radius 6 centre $(-6,0)$
2. (a) $x^2+y^2=49$
   (b) $x^2+(y-2)^2=4$
   (c) $(x-4)^2+(y+4)^2=16$
   (d) $(x+2)^2+(y+2)^2=16$
   (e) $(x+6)^2+y^2=25$
3. (a) centre $(5,-6)$ radius $\sqrt{61}$
   (b) centre $(-1,2)$ radius 4
   (c) centre (3,0) radius 5
4. (a) the region outside the circumference of the circle centre the origin radius 2.
   (b) the region inside the circle centre the origin radius 4 (often referred to as a circular disc)
   (c) the annular ring between these two circles.
5. $(x+4)^2+(y-2)^2>16$
6. $(x+2)^2+(y+1)^2<6$
7. $(x-3)^2+(y-4)^2=10$
8. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$
9. $(x-2)^2+(y-5)^2=25$ (Note: since we are told the circle touches the $x-$axis at (2,0) the centre of the circle must be at the point $(2,y_0)$ where $y_0=R$).

Some Common Functions

2.7

Introduction

This Section provides a catalogue of some common functions often used in Science and Engineering. These include polynomials, rational functions, the modulus function and the unit step function. Important properties and definitions are stated. This Section can be used as a reference when the need arises. There are, of course, other types of function which arise in engineering applications, such as trigonometric, exponential and logarithm functions. These others are dealt with in HELM 4 to HELM 6.

Prerequisites

Before starting this Section you should ...

- understand what is meant by a function and use functional notation
- be able to plot graphs of functions

Learning Outcomes

On completion you should be able to ...

- state what is meant by a polynomial function, and a rational function
- use and graph the modulus function
- use and graph the unit step function

1. Polynomial functions

A very important type of function is the **polynomial**. Polynomial functions are made up of multiples of non-negative whole number powers of a variable, such as $3x^2$, $-7x^3$ and so on. You are already familiar with many such functions. Other examples include:

$P_0(t) = 6$
$P_1(t) = 3t + 9$ (The linear function you have already met).
$P_2(x) = 3x^2 - x + 2$
$P_4(z) = 7z^4 + z^2 - 1$

where t, x and z are independent variables.

Note that fractional and negative powers of the independent variable are not allowed so that $f(x) = x^{-1}$ and $g(x) = x^{3/2}$ are not polynomials. The function $P_0(t) = 6$ is a polynomial - we can regard it as $6t^0$.

By convention a polynomial is written with the powers either increasing or decreasing. For example the polynomial

$$3x + 9x^2 - x^3 + 2$$

would be written as

$$-x^3 + 9x^2 + 3x + 2 \qquad \text{or} \qquad 2 + 3x + 9x^2 - x^3$$

In general we have the following definition:

Key Point 11

A **polynomial expression** has the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

where n is a non-negative integer, $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants and x is a variable.

A **polynomial function** $P(x)$ has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$$

The **degree** of a polynomial or polynomial function is the value of the highest power. Referring to the examples listed above, polynomial P_2 has degree 2, because the term with the highest power is $3x^2$, P_4 has degree 4, P_1 has degree 1 and P_0 has degree 0. Polynomials with low degrees have special names given in Table 5.

Table 5

	degree	name
a	0	constant
$ax + b$	1	linear
$ax^2 + bx + c$	2	quadratic
$ax^3 + bx^2 + cx + d$	3	cubic
$ax^4 + bx^3 + cx^2 + dx + e$	4	quartic

Typical graphs of some polynomial functions are shown in Figure 30. In particular, observe that the graphs of the linear polynomials, P_1 and Q_1 are straight lines.

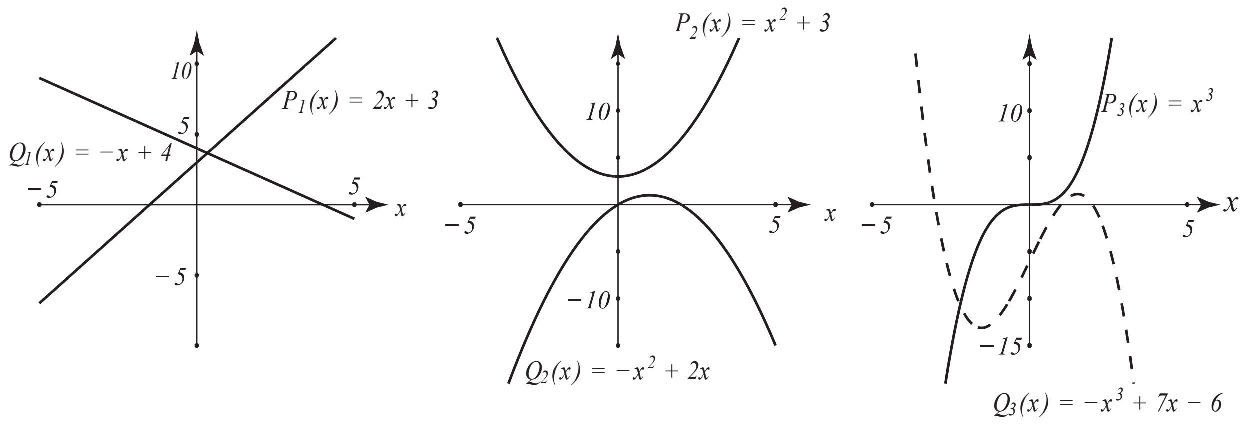


Figure 30: Graphs of some typical linear, quadratic and cubic polynomials

Task

Which of the polynomial graphs in Figure 30 are odd and which are even? Are any periodic ?

Your solution

Answer

P_2 is even. P_3 is odd. None are periodic.

Task

State which of the following are polynomial functions. For those that are, state the degree and name.

(a) $f(x) = 6x^2 + 7x^3 - 2x^4$ (b) $f(t) = t^3 - 3t^2 + 7$

(c) $g(x) = \dfrac{1}{x^2} + \dfrac{3}{x}$ (d) $f(x) = 16$ (e) $g(x) = \frac{1}{6}$

Your solution

Answer

(a) polynomial of degree 4 (quartic), (b) polynomial of degree 3 (cubic), (c) not a polynomial,

(d) polynomial of degree 0 (constant), (e) polynomial of degree 0 (constant)

Exercises

1. Write down a polynomial of degree 3 with independent variable t.
2. Write down a function which is not a polynomial.
3. Explain why $y = 1 + x + x^{1/2}$ is not a polynomial.
4. State the degree of the following polynomials: (a) $P(t) = t^4 + 7$, (b) $P(t) = -t^3 + 3$, (c) $P(t) = 11$, (d) $P(t) = t$
5. Write down a polynomial of degree 0 with independent variable z.
6. Referring to Figure 27, state which functions are one-to-one and which are many-to-one.

Answers

1. For example $f(t) = 1 + t + 3t^2 - t^3$.
2. For example $y = \dfrac{1}{x}$.
3. A term such as $x^{1/2}$, with a fractional index, is not allowed in a polynomial.
4. (a) 4, (b) 3, (c) 0, (d) 1.
5. $P(z) = 13$, for example.
6. P_1, Q_1 and P_3 are one-to-one. The rest are many-to-one.

2. Rational functions

A rational function is formed by dividing one polynomial by another. Examples include

$$R_1(x) = \frac{x+6}{x^2+1}, \qquad R_2(t) = \frac{t^3-1}{2t+3}, \qquad R_3(z) = \frac{2z^2+z-1}{z^2+z-2}$$

For convenience we have labelled these rational functions R_1, R_2 and R_3.

Key Point 12

A **rational function** has the form

$$R(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions.

P is called the **numerator** and Q is called the **denominator**.

The graphs of rational functions can take a variety of different forms and can be difficult to plot by hand. Use of a graphics calculator or computer software can help. If you have access to a plotting package or calculator it would be useful to obtain graphs of these functions for yourself. The next Example and two Tasks allow you to explore some of the features of the graphs.

Example 17

Given the rational function $R_1(x) = \dfrac{x+2}{x^2+1}$ and its graph shown in Figure 31 answer the following questions.

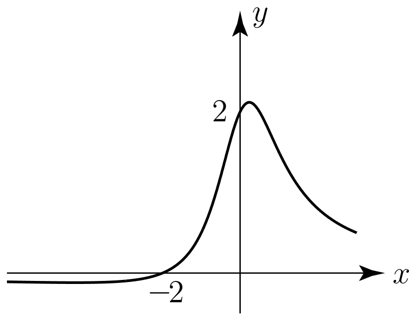


Figure 31: Graph of $R_1(x) = \dfrac{x+2}{x^2+1}$

(a) For what values of x, if any, is the denominator zero?

(b) For what values of x, if any, is the denominator negative?

(c) For what values of x is the function negative?

(d) What is the value of the function when x is zero?

(e) What happens to the function as x gets larger and larger?

Solution

(a) $x^2 + 1$ is never zero

(b) $x^2 + 1$ is never negative, it is always positive

(c) only when the numerator $x + 2$ is negative which is when x is less than -2

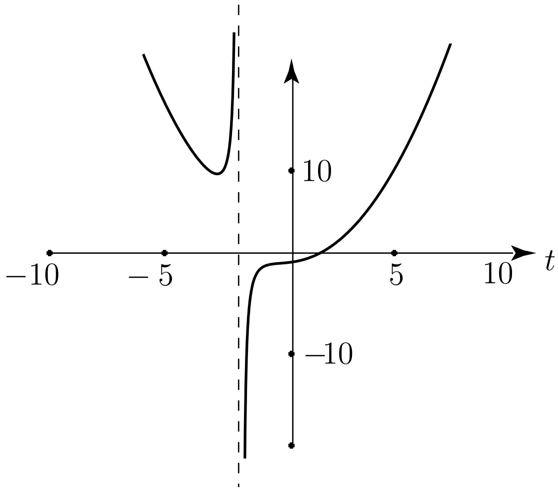
(d) 2, because that is when the numerator $x + 2 = 0$

(e) R_1 approaches zero because the x^2 term in the denominator becomes very large. (This is seen by substituting larger and larger values e.g. $10,\ 100,\ 1000\ \ldots$)

Note that for large x values the graph gets closer and closer to the x axis. We say that the x axis is a **horizontal asymptote** of this graph.

Answering questions such as (a) to (c) above will help you to sketch graphs of rational functions.

Study the graph and the algebraic form of the function $R_2(t) = \dfrac{t^3 - 1}{2t + 3}$ carefully and answer the following questions. The following figure shows its graph (the solid curve). The dotted line is an asymptote.



Graph of $R_2(t) = \dfrac{t^3 - 1}{2t + 3}$

(a) What is the function value when $t = 1$?

(b) What is the value of the denominator when $t = -3/2$?

(c) What do you think happens to the graph of the function when $t = -3/2$?

Your solution

(a)

(b)

(c)

Answer

(a) 0,

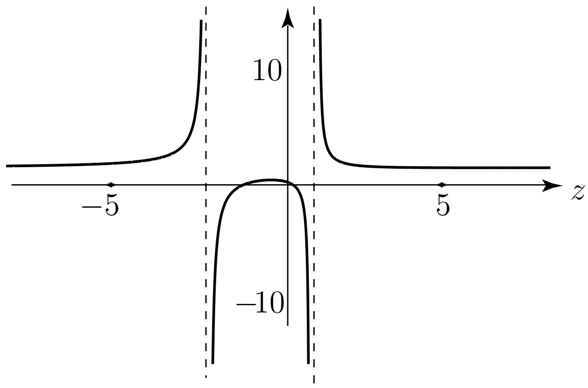
(b) 0,

(c) The function value tends to infinity, the graph becomes infinite.

Note from the answers to parts (b) and (c) that we must exclude the value $t = -3/2$ from the domain of this function because division by zero is not defined. At this point as you can see the graph shoots off towards very large positive values (we say it tends to positive infinity) if the point is approached from the left, and towards very large negative values (we say it tend to negative infinity) if the point is approached from the right. The dotted line in the graph of $R_2(x)$ has equation $t = -\frac{3}{2}$. It is approached by the curve as t approaches $-\frac{3}{2}$ and is known as a **vertical asymptote**.

Task

Study the graph and the algebraic form of the function $R_3(z) = \dfrac{2z^2 + z - 1}{(z-1)(z+2)}$ carefully and try to answer the following questions. The graph of $R_3(z)$ is shown in the following figure.



Graph of $R_3(z) = \dfrac{2z^2 + z - 1}{(z-1)(z+2)}$

(a) What is happening to the graph when $z = -2$ and when $z = 1$?
(b) Which values should be excluded from the domain of this function?
(c) Substitute some values for z (e.g. 10, 100 ...). What happens to R_3 as z gets large?
(d) Is there a horizontal asymptote?
(e) What is the name given to the vertical lines $z = 1$ and $z = -2$?

Your solution

Answer

(a) denominator is zero, R_3 tends to infinity,

(b) $z = -2$ and $z = 1$,

(c) R_3 approaches the value 2,

(d) $y = 2$ is a horizontal asymptote,

(e) vertical asymptotes

The previous Examples are intended to give you some guidance so that you will be able to sketch rational functions yourself. Each function must be looked at individually but some general guidelines are given in Key Point 13.

Key Point 13

Sketching rational functions

- Find the value of the function when the independent variable is zero. This is generally easy to evaluate and gives you a point on the graph.
- Find values of the independent variable which make the denominator zero. These values must be excluded from the domain of the function and give rise to vertical asymptotes.
- Find values of the independent variable which make the dependent variable zero. This gives you points where the graph cuts the horizontal axis (if at all).
- Study the behaviour of the function when x is large and positive and when it is large and negative.
- Are there any vertical or horizontal asymptotes? (Oblique asymptotes may also occur but these are beyond the scope of this Workbook.)

It is particularly important for engineers to find values of the independent variable for which the denominator is zero. These values are are known as the **poles** of the rational function.

Task

State the poles of the following rational functions:

(a) $f(t) = \dfrac{t-3}{t+7}$ (b) $F(s) = \dfrac{s+7}{(s+3)(s-3)}$ (c) $r(x) = \dfrac{2x+5}{(x+1)(x+2)}$

(d) $f(x) = \dfrac{x-1}{x^2-1}$.

In each case locate the poles by finding values of the independent variable which make the denominator zero:

Your solution

Answer

(a) -7, (b) 3 or -3, (c) -1 or -2, (d) $x = -1$

If you have access to a plotting package, plot these functions now.

Exercises

1. Explain what is meant by a rational function.
2. State the degree of the numerator and the degree of the denominator of the rational function $R(x) = \dfrac{3x^2 + x + 1}{x - 1}$.
3. Explain the term 'pole' of a rational function.
4. Referring to the graphs of $R_1(x), R_2(t)$ and $R_3(z)$ (on pages 66 - 68), state which functions, if any, are one-to-one and which are many-to-one.
5. Without using a graphical calculator plot graphs of $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x^2}$. Comment upon whether these graphs are odd, even or neither, whether they are continuous or discontinuous, and state the position of any poles.

Answers

1. $R(x) = P(x)/Q(x)$ where P and Q are polynomials.
2. numerator: 2, denominator: 1
3. The pole is a value of the independent variable which makes the denominator zero.
4. All are many-to-one.
5. $\dfrac{1}{x}$ is odd, and discontinuous. Pole at $x = 0$. $\dfrac{1}{x^2}$ is even and discontinuous. Pole at $x = 0$.

3. The modulus function

The modulus of a number is the size of that number with no regard paid to its sign. For example the modulus of -7 is 7. The modulus of $+7$ is also 7. We can write this concisely using the modulus sign $|\,|$. So we can write $|-7| = 7$ and $|+7| = 7$. The modulus function is defined as follows:

Key Point 14

Modulus Function

The modulus function is defined as

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

The output from the function in Key Point 14 is simply the modulus of the input.

A graph of this function is shown in Figure 32.

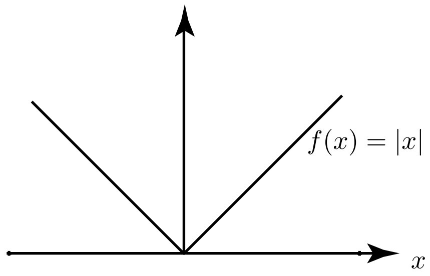


Figure 32: Graph of the modulus function $|x|$

Task

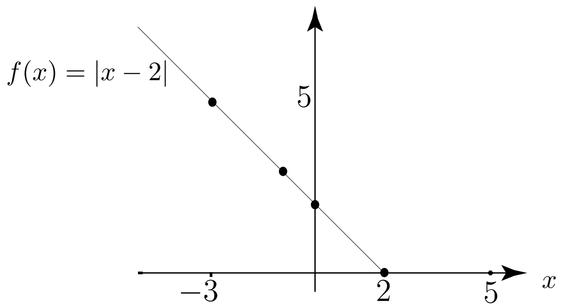
Draw up a table of values of the function $f(x) = |x - 2|$ for values of x between -3 and 5. Sketch a graph of this function.

Your solution

The table has been started. Complete it for yourself.

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	5		3	2		0			

Some points on the graph are shown in the figure. Plot your calculated points on the graph.



Answer

x	-3	-2	-1	0	1	2	3	4	5
$f(x)$	5	4	3	2	1	0	1	2	3

Exercises

1. Sketch a graph of the following functions:

   (a) $f(x) = 3|x|$, (b) $f(x) = |x+1|$, (c) $f(x) = 7|x-3|$.

2. Is the modulus function one-to-one or many-to-one?

Answers 2. Many-to-one

4. The unit step function

The unit step function is defined as follows:

Key Point 15

The unit step function $u(t)$ is defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Study this definition carefully. You will see that it is defined in two parts, with one expression to be used when t is greater than or equal to 0, and another expression to be used when t is less than 0. The graph of this function is shown in Figure 33. Note that the part of $u(t)$ for which $t < 0$ lies on the t-axis but, for clarity, is shown as a distinct dashed line.

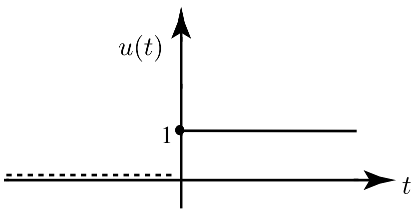


Figure 33: Graph of the unit step function

There is a jump, or discontinuity in the graph when $t = 0$. That is why we need to define the function in two parts; one part for when t is negative, and one part for when t is non-negative. The point with coordinates (0,1) is part of the function defined on $t \geq 0$.

The position of the discontinuity may be shifted to the left or right. The graph of $u(t-d)$ is shown in Figure 34.

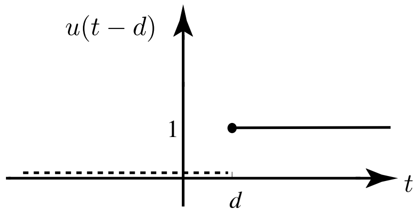


Figure 34: Graph of $u(t-d)$.

In the previous two figures the function takes the value 0 or 1. We can adjust the value 1 by multiplying the function by any other number we choose. The graph of $2u(t-3)$ is shown in Figure 35.

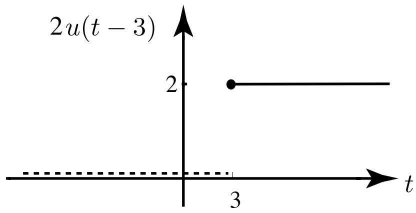


Figure 35: Graph of $2u(t-3)$

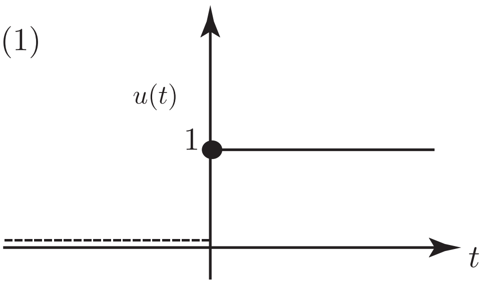
Exercises

Sketch graphs of the following functions:

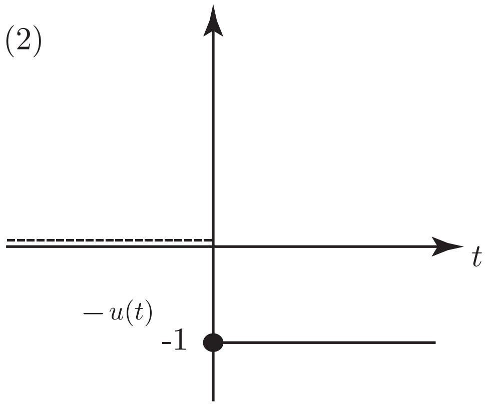
1. $u(t)$,
2. $-u(t)$,
3. $u(t-1)$,
4. $u(t+1)$,
5. $u(t-3) - u(t-2)$,
6. $3u(t)$,
7. $-2u(t-3)$.

Answers

(1)



(2)



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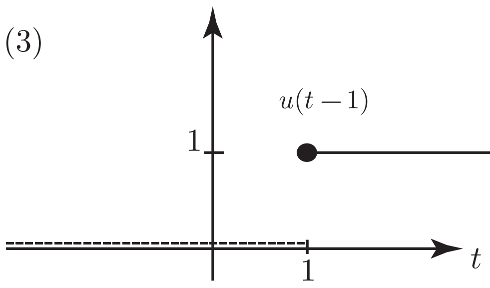
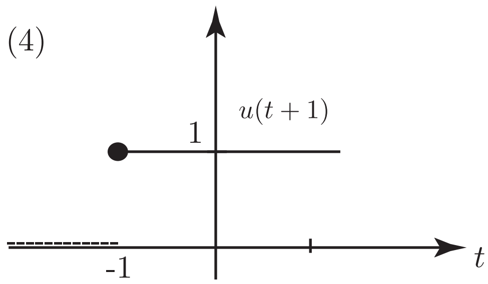


(4)



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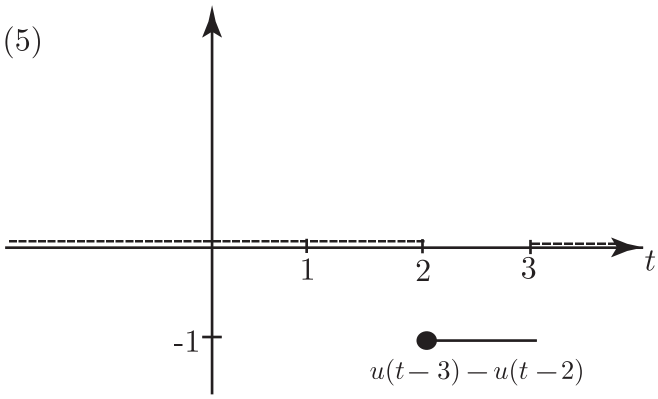

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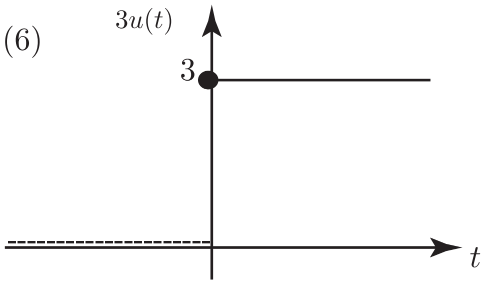

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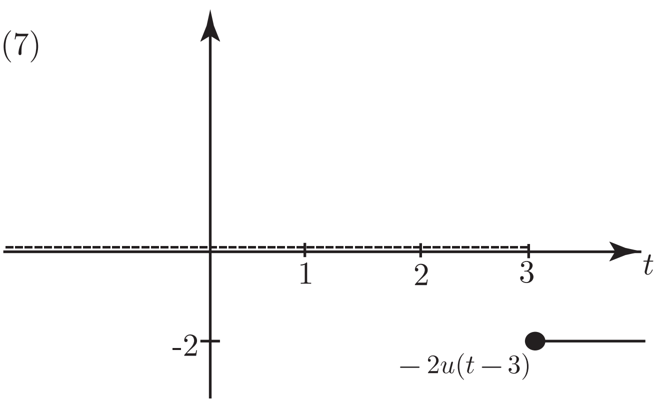

NOTES

Index for Workbook 2

Workbook 2

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